Notes on

Continuum hard-core gas models

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Abstract

Here are some notes I prepared back in 2011–2012 while learning about the setting of equilibrium statistical mechanics (in particular, hard-core gas models) in the continuum. This was the background of a project with Roberto Fernández and Santiago Saglietti, in which we (unsuccessfully) attempted to make connections between phase transitions in a continuum model and its discretized versions. (*Does the multiplicity of Gibbs measures in a continuum model imply the multiplicity of Gibbs measures in its sufficiently fine discretized versions?*) See R. Fernández, P. Groisman, S. Saglietti (Reviews in Mathematical Physics, 2016) for some related results.

Summary

Let $\mathbb{K}$ be a locally compact complete separable metric space (e.g., $\mathbb{R}^d$). A particle configuration on $\mathbb{K}$ consists of countably many particles on $\mathbb{K}$ with the condition that every bounded set contains no more than a finite number of particles. We see a particle configuration ξ as a a Radon measure on $\mathbb{K}$: for every measurable $B \subseteq \mathbb{K}$, $\xi(B)$ is the number of particles in B. (We allow multiple particles at a single point.)

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We are interested in random particle configurations, or equivalently, probability measures on the space of particle configurations. If λ is a Radon measure on $\mathbb{K}$ (e.g., the Lebesgue measure if $\mathbb{K} = \mathbb{R}^d$), a Poisson random configuration on $\mathbb{K}$ is a random particle configuration $\boldsymbol{\xi}$ such that

- for every bounded measurable subset $B \subseteq \mathbb{K}$, the distribution of $\boldsymbol{\xi}(B)$ has a Poisson distribution with intensity $\lambda(B)$, and
- for every disjoint bounded measurable subsets $B_1, B_2, \ldots, B_n \subseteq \mathbb{K}$, the random variables $\boldsymbol{\xi}(B_i)$ are independent.

A hard-core gas model is specified by conditioning a Poisson measure on a set of valid configurations.

More generally, we may have a finite set S of symbols or types, and consider particle configurations in which each particle is given a type from S. This will be identified by a tuple $(\xi^s)_{s\in S}$, where each ξ^s is an untyped particle configuration.

1 The Underlying Space

Let $\mathbb{K}$ be a locally compact separable space having a complete metric ρ. For example, $\mathbb{K}$ could be $\mathbb{R}^d$ or $\mathbb{Z}^d$.

Notation: for $a \in \mathbb{K}$ and $\varepsilon > 0$, we write $N_\varepsilon(a)$ for the open ball of radius ε around a. If $B \subseteq \mathbb{K}$, we write $N_\varepsilon(B) = \bigcup_{a\in B} N_\varepsilon(a)$ for the set of point that have distance less than ε from B.

1.1 Few facts about such a space $\mathbb{K}$. By a bounded subset of $\mathbb{K}$ we mean a set that is included in a compact subset of $\mathbb{K}$.

i) $\mathbb{K}$ has a countable base of bounded neighbourhoods.

⌜ *Argument.* Let $M \subseteq \mathbb{K}$ be a countable dense set. Each $a \in M$ has a compact neighbourhood E_a. The intersections of the interior of E_a and the open balls $N_{1/n}(a)$ for $a \in M$ and $n = 1, 2, \ldots$ form a countable base consisting of bounded open sets. ⌞

ii) $\mathbb{K}$ is σ-compact (i.e., a countable union of compact sets).

iii) For every compact $C \subseteq \mathbb{K}$, there is an open $D \supseteq C$ whose closure $\overline{D}$ is compact.

⌜ *Argument.* For each $a \in C$ let E_a be a bounded open neighbourhood of a. Then $\{E_a\}_{a\in C}$ is an open cover of C, and has a finite sub-cover $\{E_a\}_{a\in I}$. The set $\bigcup_{a\in I} E_a$ is open and bounded (because it is a finite union of bounded sets), and it includes C. ⌞

1.2 Some classes of functions.

$C(\mathbb{K})$ set of continuous functions $f : \mathbb{K} \to \mathbb{R}$.

$C_c(\mathbb{K})$ set of compactly supported continuous functions $f : \mathbb{K} \to \mathbb{R}$. (The support of f, denoted by $\text{supp}(f)$ is the smallest closed set C with $f(a) = 0$ for every $a \notin C$)

$C_\circ(\mathbb{K})$ set of continuous functions $f : \mathbb{K} \to \mathbb{R}$ that vanish at infinity (i.e., $\{a : |f(a)| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$).

$BC(\mathbb{K})$ set of bounded continuous functions $f : \mathbb{K} \to \mathbb{R}$.

The default topology on each of these is the topology of the uniform norm. We have

$$C_c(\mathbb{K}) \subseteq \overline{C_c(\mathbb{K})} = C_o(\mathbb{K}) \subseteq BC(\mathbb{K}) \subseteq C(\mathbb{K}) . \tag{1}$$

1.3 Separability of $C_c(\mathbb{K})$. The set $C_c(\mathbb{K})$ has a countable dense subset.

⌜ *Argument.* This is well-known to be true if $\mathbb{K}$ is compact: it follows from the Stone-Weierstrass theorem (e.g., Theorem 44.5 of [16]) and the separability of $\mathbb{K}$. If $\mathbb{K}$ is not compact, let $U_1 \subseteq U_2 \subseteq \cdots$ be bounded open sets with $\mathbb{K} = \bigcup_{i\geq 1} U_i$ (see 1.1). For each i, choose a countable dense subset D_i of continuous functions whose support is included in U_i. Then, $\bigcup_{i\geq 1} D_i$ is a countable dense subset of $C_c(\mathbb{K})$. ⌞

Moreover, every dense $F(\mathbb{K}) \subseteq C_c(\mathbb{K})$ has a countable dense subset.

⌜ *Argument.* Since separable and metric, $C_c(\mathbb{K})$ has a countable base $\mathscr{B}$. From each $B \in \mathscr{B}$, pick $a \in B \cap F(\mathbb{K})$. ⌞

Let us say that a set $F(\mathbb{K}) \subseteq C_c(\mathbb{K})$ is properly dense if for each $f \in C_c(\mathbb{K})$ and every $\varepsilon > 0$ there is $g \in F(\mathbb{K})$ such that $\|f - g\| < \varepsilon$ and $\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$, and furthermore, the function g can be chosen to be non-negative if f is non-negative. There exist a countable properly dense subset of $C_c(\mathbb{K})$.

⌜ *Argument.* Let $\mathscr{B} = \{B_0, B_1, \ldots\}$ be a countable base of $\mathbb{K}$ such that $\overline{B}_i$ are compact (see 1.1). For each finite $I \subseteq \mathbb{N}$, let $B_I \triangleq \bigcup_{i\in I} B_i$ and choose a countable dense subset F_I of continuous functions supported at B_I. Set $\tilde{F} \triangleq \bigcup_{\substack{I\subseteq\mathbb{N}\\ \text{finite}}} F_I$. We claim that $F \triangleq \{g, |g| : g \in \tilde{F}\}$ is properly dense.

Let $f \in C_c(\mathbb{K})$ and $\varepsilon > 0$. Let $A_\varepsilon \triangleq \{a \in \mathbb{K} : |f(a)| \geq \varepsilon/2\}$. Then, A_ε is included in the interior of $\mathrm{supp}(f)$. For every $x \in A_\varepsilon$, there is k such that $x \in B_k \subseteq \mathrm{supp}(f)$. By compactness, there is a finite index set $I \subseteq \mathbb{N}$ such that $A_\varepsilon \subseteq B_I \subseteq \mathrm{supp}(f)$. Let $h_\varepsilon : \mathbb{K} \to [0, 1]$ be a continuous function with

$$h_\varepsilon(a) = \begin{cases} 1 & \text{if } a \in A_\varepsilon, \\ 0 & \text{if } a \notin B_I. \end{cases} \tag{2}$$

(Such a function exists by Urysohn's lemma.) Choose $g_\varepsilon \in F_I$ with $\|g_\varepsilon - fh_\varepsilon\| < \varepsilon/2$. Then, $\mathrm{supp}(g_\varepsilon) \subseteq \mathrm{supp}(f)$ and $\|g_\varepsilon - f\| < \varepsilon$.

Furthermore, if f is non-negative, we also have $\mathrm{supp}(|g_\varepsilon|) = \mathrm{supp}(g_\varepsilon) \subseteq \mathrm{supp}(f)$ and $\||g_\varepsilon| - f\| \leq \|g_\varepsilon - f\| < \varepsilon$. ⌞

1.4 Approximating sets by functions. Every compact set (resp., bounded open set) is a pointwise monotone limit of elements of $C_c(\mathbb{K})$:

- For every compact set $V \subseteq \mathbb{K}$, there is a decreasing sequence $g_1, g_2, \ldots \in C_c(\mathbb{K})$ such that $g_n \searrow 1_V$ pointwise.

⌜ *Argument.* Let $A_1, A_2, \ldots \subseteq \mathbb{K}$ be a sequence of open sets with compact closure such that $A_n \supseteq \overline{A}_{n+1}$ for every n, and $\bigcap_n A_n = V$. (Simply, let $A \supseteq V$ be a bounded open set (see 1.1), and set $A_n \triangleq A \cap N_{1/n}(V)$, where $N_{1/n}(V)$ is the set of points within distance less than $1/n$ from V.) By Urysohn's lemma, there are continuous functions $g_n : \mathbb{K} \to [0, 1]$ such that

$$g_n(a) = \begin{cases} 1 & \text{if } a \in \overline{A}_{n+1}, \\ 0 & \text{if } a \notin A_n. \end{cases} \tag{3}$$

Then, $g_n \geq g_{n+1}$ and $g_n(a) = 1_V(a)$ for every $a \notin A_n \setminus V$. ⌞

- For every bounded open set $U \subseteq \mathbb{K}$, there is an increasing sequence $h_1, h_2, \ldots \in C_c(\mathbb{K})$ such that $h_n \nearrow 1_U$ pointwise.

The first approximation above remains valid if we require the approximating functions to be chosen from a properly dense subset (see 1.3). Let $F(\mathbb{K})$ be a properly dense subset of $C_c(\mathbb{K})$. For every compact $V \subseteq \mathbb{K}$, there is a decreasing sequence $h_1, h_2, \ldots \in F(\mathbb{K})$ such that $h_n \searrow 1_V$ pointwise.

Argument. As before, let $A_1, A_2, \ldots \subseteq \mathbb{K}$ be a sequence of open sets with compact closure such that $A_n \supseteq \overline{A}_{n+1}$ for every n, and $\bigcap_n A_n = V$. For each n, let $g_n : \mathbb{K} \to [0, 1 + 2^{-n}]$ be a continuous function, provided by Urysohn's lemma, such that

$$g_n(a) = \begin{cases} 1 + 2^{-n} & \text{if } a \in \overline{A}_{n+1}, \\ 0 & \text{if } a \notin A_n \end{cases} \tag{4}$$

Choose $h_n \in F(\mathbb{K})$ such that $h_n \geq 0$, $\operatorname{supp}(h_n) \subseteq \operatorname{supp}(g_n)$ and $\|g_n - h_n\| < 2^{-n-2}$.

1.5 The Borel σ-algebra on $\mathbb{K}$. Let $\mathscr{E} \subseteq 2^{\mathbb{K}}$ be the class of Borel-measurable bounded subsets of $\mathbb{K}$. Then, $\mathscr{E}$ is a ring (i.e., $\varnothing \in \mathscr{E}$, and $A, B \in \mathscr{E}$ implies $A \cup B, A \setminus B \in \mathscr{E}$). In particular,

$$\widehat{\mathscr{E}} \triangleq \{A, \mathbb{K} \setminus A : A \in \mathscr{E}\} \tag{5}$$

is an algebra (i.e., $\varnothing \in \widehat{\mathscr{E}}$, and $A, B \in \widehat{\mathscr{E}}$ implies $\mathbb{K} \setminus A, A \cup B, A \cap B \in \widehat{\mathscr{E}}$). Since $\mathbb{K}$ has a countable base of bounded sets, the family $\mathscr{E}$ generates the Borel σ-algebra on $\mathbb{K}$.

By Carathéodory's extension theorem (e.g., Theorem 3.1.4 of [2]), every countably additive function $\mu : \mathscr{E} \to [0, \infty]$ with $\mu(\varnothing) = 0$ extends to a Borel measure. Furthermore, if μ is finite on $\mathscr{E}$, the extension is unique.

If $\mathbb{K} = \mathbb{R}^d$, we could also work with the ring generated by half-open half-closed hypercubes $[a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_d, b_d)$ for $a_i, b_i \in \mathbb{R}$. The collection of such hypercubes forms a semi-ring $\mathscr{E}_\circ$ (i.e., $\varnothing \in \mathscr{E}_\circ$, and $A, B \in \mathscr{E}_\circ$ implies $A \cap B \in \mathscr{E}_\circ$ and $A \setminus B = \bigcup_{i=1}^{n} C_i$ for some disjoint $C_1, C_2, \ldots, C_n \in \mathscr{E}_\circ$) and generates the Borel σ-algebra on $\mathbb{R}^d$. A similar extension property for countably additive functions on $\mathscr{E}_\circ$ holds.

1.6 Radon measures on $\mathbb{K}$. A Radon measure on $\mathbb{K}$ is a Borel measure μ with $\mu(C) < \infty$ for every compact set $C \subseteq \mathbb{K}$. Every Radon measure is uniquely determined by its values on bounded sets (see 1.5).

We call a Borel measure μ on $\mathbb{K}$ regular if

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\} \tag{6}$$

$$= \sup\{\mu(V) : \text{compact } V \subseteq E\}\ . \tag{7}$$

Note the difference with the other common definition of regularity in which V (in the second equality) is only required to be closed.

Every Radon measure on $\mathbb{K}$ is regular (e.g., Theorem 7.8 of [4]). This follows from Ulam's theorem (Theorem 7.1.4 of [2]), which states that every finite Borel measure on a complete separable metric space is regular.

1.7 Particle configurations on $\mathbb{K}$. A particle configuration on $\mathbb{K}$ is a Radon measure ξ such that $\xi(B) \in \mathbb{N}$ for every bounded measurable $B \subseteq \mathbb{K}$.

Let $Q \subseteq \mathbb{K}$ be a countable set such that for every compact $C \subseteq \mathbb{K}$, the set $Q \cap C$ is finite. Let $n : Q \to \mathbb{N} \setminus \{0\}$. Then, $\xi \triangleq \sum_{a \in Q} n(a)\delta_a$ is a particle configuration on $\mathbb{K}$.

Conversely, let ξ be a particle configuration on $\mathbb{K}$. For each $a \in \mathbb{K}$, define $n(a) \triangleq \xi(\{a\})$, and set $Q \triangleq \{a : n(a) > 0\}$. For every compact $C \subseteq \mathbb{K}$, we have $|Q \cap C| \leq \xi(C) < \infty$. This also implies that Q is countable, because $\mathbb{K}$ is a countable union of compact sets. We have $\xi = \sum_{a \in Q} n(a)\delta_a$.

⌜ *Argument.* Let $B \subseteq \mathbb{K}$ be a bounded measurable set. Then, $\xi(B) \geq \sum_{a \in Q \cap B} \xi(\{a\}) = \sum_{a \in Q} n(a)\delta_a(B)$. If $\xi(B) > \sum_{a \in Q \cap B} \xi(\{a\})$, by regularity of ξ, there is a compact set $C_0 \subseteq B \setminus Q$ such that $\xi(C_0) \geq 1$. Let $A_1, A_2, \ldots, A_m$ be an open cover of C_0 with balls with diameter at most 2^{-1}. Then, there must be i such that $\xi(A_i \cap C_0) \geq 1$. By regularity, there is a compact set $C_1 \subseteq A_i \cap C_0$ with $\xi(C_1) \geq 1$. Similarly, we can find a chain $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$ of compact sets such that C_n has diameter at most 2^{-n} and $\xi(C_n) \geq 1$. The intersection $\bigcap_n C_n$ contains a single point x with $\xi(\{x\}) \geq 1$, contradicting the fact that $C_0 \cap Q = \varnothing$. ⌞

We call $\xi = \sum_{a \in Q} n(a)\delta_a$ the standard representation of ξ.

1.8 Radon measures as linear functionals. Every compactly supported continuous function $f : \mathbb{K} \to \mathbb{R}$ is integrable with respect to any Radon measure on $\mathbb{K}$. Note, however, that an element of $C_\circ(\mathbb{K})$ could be non-integrable with respect to a non-finite Radon measure.

Each Radon measure μ on $\mathbb{K}$ defines a positive linear functional $f \mapsto \mu(f) = \int f \mathrm{d}\mu$ on $C_\mathrm{c}(\mathbb{K})$. Moreover, μ is uniquely determined by this functional.

⌜ *Argument.* Let μ and μ' be two Radon measures that agree on $C_\mathrm{c}(\mathbb{K})$. Since $\mathscr{E}$ (the ring of bounded measurable subsets of $\mathbb{K}$) generates the Borel σ-algebra, it is enough to verify that $\mu(B) = \mu'(B)$ for each $B \in \mathscr{E}$.

Let $B \in \mathscr{E}$. Let $D \supseteq \overline{B}$ be an open set such that $\overline{D}$ is compact (see 1.1). Let $\varepsilon > 0$. By the regularity of μ and μ', there is a compact set $V \subseteq B$ and an open set $U \supseteq B$ with $U \subseteq D$ wuch that $\mu(U \setminus V), \mu'(U \setminus V) < \varepsilon/2$. By Urysohn's lemma, there is a continuous function $f_\varepsilon : \mathbb{K} \to [0,1]$ with

$$f_\varepsilon(a) = \begin{cases} 1 & \text{if } a \in V, \\ 0 & \text{if } a \notin U. \end{cases} \tag{8}$$

Since $\overline{U}$ is compact, $f_\varepsilon \in C_\mathrm{c}(\mathbb{K})$. We have

$$\mu(B) - \varepsilon/2 < \mu(V) \leq \mu(f_\varepsilon) \leq \mu(U) < \mu(B) + \varepsilon/2 \,, \tag{9}$$

$$\mu'(B) - \varepsilon/2 < \mu'(V) \leq \mu'(f_\varepsilon) \leq \mu'(U) < \mu'(B) + \varepsilon/2 \,, \tag{10}$$

which imply $|\mu(B) - \mu'(B)| < \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, we find that $\mu(B) = \mu'(B)$. ⌞

Conversely, according to the Riesz representation theorem (e.g., Theorem 7.2 of [4]) every positive linear function $J : C_\mathrm{c}(\mathbb{K}) \to \mathbb{R}$ identifies a Radon measure μ on $\mathbb{K}$ with $\mu(f) = J(f)$ for every $f \in C_\mathrm{c}(\mathbb{K})$.

2 Space of Radon Measures

Let $\mathcal{M}[\mathbb{K}]$ denote the set of Radon measures on $\mathbb{K}$. When $\mathbb{K}$ is clear from the context, we may also use a shorter name $\mathcal{M}$ instead of $\mathcal{M}[\mathbb{K}]$. The vague topology on $\mathcal{M}[\mathbb{K}]$ is the weakest topology that makes all the observations $\mu \mapsto \mu(f)$ for $f \in C_\mathrm{c}(\mathbb{K})$ continuous. In particular, $\mu_i \xrightarrow{\mathrm{v}} \mu$ (μ_i converges vaguely to μ) if and only if $\mu_i(f) \to \mu(f)$ for every $f \in C_\mathrm{c}(\mathbb{K})$.

2.1 A base for the vague topology. By definition, the sets

$$\mathcal{U}(\mu, f, \varepsilon) = \{\nu : |\mu(f) - \nu(f)|\} < \varepsilon\} \tag{11}$$

for $\mu \in \mathcal{M}[\mathbb{K}]$, $f \in C_c(\mathbb{K})$ and $\varepsilon > 0$ form a sub-base (i.e., generating set) for the vague topology on $\mathcal{M}[\mathbb{K}]$. Therefore, the family of finite intersections

$$\mathcal{U}(\mu, f_1, f_2, \ldots, f_n, \varepsilon) = \bigcap_{i=1}^{n} \mathcal{U}(\mu, f_i, \varepsilon) \tag{12}$$

for $\mu \in \mathcal{M}[\mathbb{K}]$, $f_i \in C_c(\mathbb{K})$ and $\varepsilon > 0$ is a base for the vague topology.

2.2 Set measurements.

- If $V \subseteq \mathbb{K}$ is compact, the mapping $\mu \mapsto \mu(V)$ is upper semi-continuous (i.e., for every $R > 0$, the set $\{\mu : \mu(V) < R\}$ is open).

Argument. There is a decreasing sequence $g_1, g_2, \ldots \in C_c(\mathbb{K})$ such that $g_n \searrow 1_V$ pointwise (see 1.4). By monotone continuity, for each $\mu \in \mathcal{M}[\mathbb{K}]$, we have $\mu(g_n) \searrow \mu(V)$. We have

$$\{\mu : \mu(V) < R\} = \bigcup_n \{\mu : \mu(g_n) < R\} . \tag{13}$$

- If $U \subseteq \mathbb{K}$ is open and bounded, the mapping $\mu \mapsto \mu(U)$ is lower semi-continuous (i.e., for every $R > 0$, the set $\{\mu : \mu(U) > R\}$ is open).
- If $B \subseteq \mathbb{K}$ is bounded and measurable, the mapping $\mu \mapsto \mu(B)$ is continuous at each point $\nu \in \mathcal{M}[\mathbb{K}]$ with $\nu(\partial B) = 0$.

Argument. For every $\varepsilon > 0$, the set

$$\mathcal{A} \triangleq \{\mu : \mu(\overline{B}) < \nu(B) + \varepsilon\} \cap \{\mu : \mu(\mathring{B}) > \nu(B) - \varepsilon\} \tag{14}$$

is open and contains ν. Furthermore, for every $\mu \in \mathcal{A}$, it holds $|\mu(B) - \nu(B)| < \varepsilon$.

A measurable set $B \subseteq \mathbb{K}$ is called a continuity set of a Radon measure ν if $\nu(\partial B) = 0$. If A and B are continuity sets of a Radon measure ν, so are $A \cap B$, $A \cup B$ and $\mathbb{K} \setminus A$.

2.3 Criteria for vague convergence. Let $\mu, \mu_1, \mu_2, \ldots$ be Radon measures on $\mathbb{K}$. The following conditions are equivalent (e.g., Theorem A 7.2 of [8]):

i) $\mu_n \xrightarrow{\text{v}} \mu$ (μ_n vaguely converges to μ),

ii) $\mu_n(B) \to \mu(B)$ for every bounded measurable $B \subseteq \mathbb{K}$ with $\mu(\partial B) = 0$,

iii) $\limsup \mu_n(V) \leq \mu(V)$ and $\liminf \mu_n(U) \geq \mu(U)$ for every compact $V \subseteq \mathbb{K}$ and every bounded open $U \subseteq \mathbb{K}$.

2.4 $\mathcal{M}[\mathbb{K}]$ is separable. The elements of $\mathcal{M}[\mathbb{K}]$ having compact support are dense.

Argument. Let $\mathcal{U}(\mu, f_1, f_2, \ldots, f_n, \varepsilon)$ be a neighbourhood. Set $A = \bigcup_{i=1}^{n} \operatorname{supp}(f_i)$ and $\hat{\mu}(\cdot) \triangleq \mu(\cdot \cap A)$.

For a compact set $C \subseteq \mathbb{K}$, let $\mathcal{M}[\mathbb{K} \,|\, C]$ denote the set of Radon measures whose supports are included in C. If $R \geq 0$, let $\mathcal{M}^{\leq R}[\mathbb{K} \,|\, C]$ be the set of $\mu \in \mathcal{M}[\mathbb{K} \,|\, C]$ with $\mu(C) \leq R$. The space $\mathcal{M}^{\leq R}[\mathbb{K} \,|\, C]$ is compact. We have $\mathcal{M}[\mathbb{K} \,|\, C] = \bigcup_{n=0}^{\infty} \mathcal{M}^{\leq n}[\mathbb{K} \,|\, C]$. Therefore, $\mathcal{M}[\mathbb{K} \,|\, C]$ is locally compact and σ-compact.

Let $C_1 \subseteq C_2 \subseteq \cdots$ be a sequence of compact sets with $\bigcup_{i=1}^{\infty} C_i = \mathbb{K}$. Then, for every $R \geq 0$ and $i \geq 1$, the set $\mathcal{M}^{\leq R}[\mathbb{K} \,|\, C_i]$ has a countable dense set, because it is a compact metrizable space. Furthermore, $\bigcup_{i=1}^{\infty} \bigcup_{n=0}^{\infty} \mathcal{M}^{\leq n}[\mathbb{K} \,|\, C]$ is dense in $\mathcal{M}[\mathbb{K}]$. Therefore, $\mathcal{M}[\mathbb{K}]$ has a countable dense set.

A particular countable dense set can be constructed as follows. Let $D \subseteq \mathbb{K}$ be a countable dense set. Then, the positive rational linear combinations of Dirac measures δ_a for $a \in D$ are dense in $\mathcal{M}[\mathbb{K}]$.

Argument. This is well-known to be true when restricted to $\mathcal{M}^{\leq R}[\mathbb{K} \,|\, C]$, where $R \geq 0$ and $C \subseteq \mathbb{K}$ compact.

2.5 Countable generation of vague topology. We want to show that there is a countable set $F(\mathbb{K}) \subseteq C_{\mathrm{c}}(\mathbb{K})$ such that the vague topology is generated by the mappings $\mu \mapsto \mu(g)$, for $g \in F(\mathbb{K})$. We show that a countable properly dense subset $F(\mathbb{K}) \subseteq C_{\mathrm{c}}(\mathbb{K})$ would do (see 1.3).

Let $F(\mathbb{K})$ be a properly dense subset of $C_{\mathrm{c}}(\mathbb{K})$. Let $\mathscr{T}$ denote the weakest topology on $\mathcal{M}[\mathbb{K}]$ that makes all the projections $\mu \mapsto \mu(g)$ continuous for all $g \in F(\mathbb{K})$.

For every compact set $V \subseteq \mathbb{K}$ and every $R > 0$, the set $\{\mu : \mu(V) < R\}$ is open with respect to $\mathscr{T}$.

Argument. This is similar to the vague topology (see 2.2). Let $h_1, h_2, \ldots$ be a decreasing sequence in $F(\mathbb{K})$ such that $h_n \searrow 1_V$ (see 1.4). By monotone continuity, for each $\mu \in \mathcal{M}[\mathbb{K}]$, we have $\mu(h_n) \searrow \mu(V)$. We have

$$\{\mu : \mu(V) < R\} = \bigcup_n \{\mu(h_n) < R\} \; . \tag{15}$$

The topology $\mathscr{T}$ coincides with the vague topology. Namely, for every $f \in C_{\mathrm{c}}(\mathbb{K})$, the mapping $\mu \mapsto \mu(f)$ is continuous with respect to $\mathscr{T}$.

Argument. Let $\mu_0 \in \mathcal{M}[\mathbb{K}]$ and $\varepsilon > 0$. We find a $\mathscr{T}$-open set $\mathcal{U} \subseteq \mathcal{M}[\mathbb{K}]$ with $\mu_0 \in \mathcal{U}$ such that $|\mu(f) - \mu_0(f)| < \varepsilon$ for every $\mu \in \mathcal{U}$.

Let $V \triangleq \operatorname{supp}(f)$. Choose $R > 0$ with $\mu_0(V) < R$, and $g_\varepsilon \in F(\mathbb{K})$ with $\operatorname{supp}(g_\varepsilon) \subseteq V$ and $\|g_\varepsilon - f\| < \varepsilon/(3R)$. Set

$$\mathcal{U} \triangleq \{\mu : |\mu(g_\varepsilon) - \mu_0(g_\varepsilon)| < \varepsilon/3\} \cap \{\mu : \mu(V) < R\} \tag{16}$$

This is open with respect to $\mathscr{T}$. By construction, $\mu_0 \in U$. For $\mu \in \mathcal{U}$ we have

$$\begin{aligned} |\mu(f) - \mu_0(f)| &< |\mu(f) - \mu(g_\varepsilon)| + |\mu(g_\varepsilon) - \mu_0(g_\varepsilon)| + |\mu_0(g_\varepsilon) - \mu_0(f)| && (17)\\ &< \|f - g_\varepsilon\| \, R + \varepsilon/3 + \|f - g_\varepsilon\| \, R && (18)\\ &< \varepsilon \; . && (19) \end{aligned}$$

2.6 The vague topology on $\mathcal{M}[\mathbb{K}]$ is metric. (e.g., Section A 7.7 of [8]) Let $g_1, g_2, \ldots$ be a properly dense sequence in $C_c(\mathbb{K})$ (see 1.3). The vague topology is the weakest topology that makes all the mappings $\mu \mapsto \mu(g_k)$, for $k = 1, 2, \ldots$, continuous (see 2.5).

$$\rho_{\mathcal{M}}(\mu, \nu) \triangleq \sum_{k=1}^{\infty} 2^{-k} \left(1 - \mathrm{e}^{-|\mu(g_k) - \nu(g_k)|}\right) \tag{20}$$

is a metric on $\mathcal{M}[\mathbb{K}]$ that generates the vague topology.

2.7 Criterion for vague compactness. Let $R : C_c(\mathbb{K}) \to \mathbb{R}^{\geq 0}$ be given. The set

$$\mathcal{Q}_R \triangleq \{\mu \in \mathcal{M}[\mathbb{K}] : |\mu(f)| \leq R(f) \text{ for all } f \in C_c(\mathbb{K})\} \tag{21}$$

is compact.

Proof. Let $\mu_1, \mu_2, \ldots$ be a sequence in $\mathcal{Q}_R$. We want to show that it has a converging subsequence.

Pick a countable properly dense set $F(\mathbb{K})$ of $C_c(\mathbb{K})$ (see 1.3). Let $g_1, g_2, \ldots$ be an enumeration of the elements of $F(\mathbb{K})$. Since $\{\mu_n(g_1)\}_n$ is bounded by $R(g_1)$, there is a subsequence $\{n(1,i)\}_i$ of $\{n\}_n$ such that the limit $\tilde{\mu}(g_1) \triangleq \lim_{i\to\infty} \mu_{n(1,i)}(g_1)$ exists and is bounded by $R(g_1)$. Inductively, for each $k > 1$, since $\{\mu_n(g_k)\}_n$ is bounded by $R(g_k)$, there is a subsequence $\{n(k,i)\}_i$ of $\{n(k-1,i)\}_i$ such that the limit $\tilde{\mu}(g_k) \triangleq \lim_{i\to\infty} \mu_{n(k,i)}(g_k)$ exists and is bounded by $R(g_k)$.

Then, for each k, the diagonal subsequence $\{n(i,i)\}_i$ is eventually a subsequence of $\{n(k,i)\}_i$. Therefore, $\tilde{\mu}(g) = \lim_{i\to\infty} \mu_{n(i,i)}(g) \leq R(g)$ for each $g \in F(\mathbb{K})$. We claim that for $f \in C_c(\mathbb{K})$, the limit $\tilde{\mu}(f) \triangleq \lim_{i\to\infty} \mu_{n(i,i)}(f)$ exists and is bounded by $R(f)$.

⌜ *Argument.* Let $\varepsilon > 0$ and $V \triangleq \mathrm{supp}(f)$. Pick $h \in F(\mathbb{K})$ with $h \geq 1_V$ (existence follows e.g. using 1.4). Then, $\mu(V) \leq \mu(h) \leq R(h)$ for all $\mu \in \mathcal{Q}_R$. Pick $g_\varepsilon \in F(\mathbb{K})$ with $\|f - g_\varepsilon\| < \varepsilon/R(h)$ and $\mathrm{supp}(g_\varepsilon) \subseteq V$. Then, for all $\mu \in \mathcal{Q}_R$ we have $|\mu(f) - \mu(g_\varepsilon)| \leq \|f - g_\varepsilon\|\, \mu(V) < \varepsilon$.

Therefore, $\{\mu_{n(i,i)}(f)\}_i$ is ε-close to the convergent sequence $\{\mu_{n(i,i)}(g_\varepsilon)\}_i$. Since this is true for every $\varepsilon > 0$, we obtain that $\{\mu_{n(i,i)}(f)\}_i$ is Cauchy, hence convergent.

⌞ The limit $\tilde{\mu}(f) \triangleq \lim_{i\to\infty} \mu_{n(i,i)}(f)$ is clearly bounded by $R(f)$.

The mapping $\tilde{\mu} : C_c(\mathbb{K}) \to \mathbb{R}$ is positive linear. Therefore, by Riesz's theorem, it defines a Radon measure (see 1.8). □

If $F(\mathbb{K})$ is a properly dense subset of $C_c(\mathbb{K})$ (see 1.3), for every $R : F(\mathbb{K}) \to \mathbb{R}^{\geq 0}$, the set

$$\mathcal{Q}'_R \triangleq \{\mu \in \mathcal{M}[\mathbb{K}] : |\mu(f)| \leq R(f) \text{ for all } f \in F(\mathbb{K})\} \tag{22}$$

is also compact.

⌜ *Argument.* We find $R' : C_c(\mathbb{K}) \to \mathbb{R}^{\geq 0}$ such that $\mathcal{Q}'_R = \mathcal{Q}_{R'}$.

For each $g \in F(\mathbb{K})$, set $R'(g) \triangleq R(g)$. Let $f \in C_c(\mathbb{K})$ and $V \triangleq \mathrm{supp}(f)$. Pick $h \in F(\mathbb{K})$ with $h \geq 1_V$ (existence follows e.g. using 1.4). Then, $\mu(V) \leq \mu(h) \leq R(h)$ for all $\mu \in \mathcal{Q}_R$.

Pick an arbitrary $\varepsilon > 0$. Choose $g \in F(\mathbb{K})$ with $\mathrm{supp}(g) \subseteq V$ such that $\|g - f\| < \varepsilon$. Then,
⌞ $|\mu(f) - \mu(g)| \leq \|f - g\|\, \mu(V) \leq \varepsilon R(h)$. Set $R(f) \triangleq R(g) + \varepsilon R(h)$.

Let $\mathcal{D} \subseteq \mathcal{M}[\mathbb{K}]$. Then $\mathcal{D}$ has compact closure if and only if for every bounded $B \subseteq \mathbb{K}$ (or for every compact $B \subseteq \mathbb{K}$) it holds $\sup\{\mu(B) : \mu \in \mathcal{D}\} < \infty$ (e.g., Theorem A 7.5 in [8]).

Argument. First, suppose that $\{\mu(B) : \mu \in \mathcal{D}\}$ is not bounded. We can assume that B is compact, for otherwise $\overline{B}$ has the same property. Choose $\mu_1, \mu_2, \ldots \in \mathcal{D}$ such that $\mu_n(B) \nearrow \infty$. If $n_1 < n_2 < \cdots$ is any subsequence, we have $\mu_{n_i}(B) \nearrow \infty$. Therefore, $\{\mu_n\}_n$ has no converging subsequence (see 2.3), and hence the closure of $\mathcal{D}$ is not compact.

Next suppose that for every compact $B \subseteq \mathbb{K}$, we have $R_0(B) \triangleq \sup\{\mu(B) : \mu \in \mathcal{D}\} < \infty$. Then for every $f \in C_c(\mathbb{K})$, if we let $R(f) \triangleq \|f\| R_0(\mathrm{supp}(f))$, we have $\mu(f) \leq \|f\| \mu(\mathrm{supp}(f)) \leq R(f)$. Therefore, $\mathcal{D} \subseteq \mathcal{Q}_R$. Since $\mathcal{Q}_R$ is compact, we conclude that the closure of $\mathcal{D}$ is also compact.

2.8 The vague topology on $\mathcal{M}[\mathbb{K}]$ has a complete metric. Let $\rho_{\mathcal{M}}$ be the metric defined in 2.6. Let $\mu_1, \mu_2, \ldots$ be a sequence in $C_c(\mathbb{K})$ that is Cauchy with respect to $\rho_{\mathcal{M}}$. Then, for each k, the sequence $\mu_1(g_k), \mu_2(g_k), \ldots$ is Cauchy, hence bounded. Set $R(g_k) \triangleq \sup_n |\mu_n(g_k)|$. Then, $\{\mu_n\}_n$ lies in the set

$$\mathcal{Q}'_R \triangleq \{\mu : |\mu(g_k)| \leq R(g_k) \text{ for } k = 1, 2, \ldots\} \tag{23}$$

which is compact (see 2.7).

3 Space of Particle Configurations

Let $\mathcal{N}[\mathbb{K}]$ denote the set of particle configurations on $\mathbb{K}$ (see 1.7). When $\mathbb{K}$ is clear from the context, we may also use a shorter name $\mathcal{N}$ instead of $\mathcal{N}[\mathbb{K}]$.

Notation: if ξ is a particle configuration and $C \subseteq \mathbb{K}$ a measurable set, let us write $\xi_C \triangleq \xi(\cdot \cap C)$. This is seen as the restriction of the configuration ξ to C, or the projection of ξ on C.

3.1 $\mathcal{N}[\mathbb{K}]$ is vaguely closed in $\mathcal{M}[\mathbb{K}]$. (Proposition 2.2 in [7] or Proposition A 7.4 in [8])

3.2 Relative vague topology. Two remarks:

- If $V \subseteq \mathbb{K}$ is compact and $n \in \mathbb{N}$, the set

$$\{\xi : \xi(V) \leq n\} = \{\xi : \xi(V) < n + 1\} \tag{24}$$

is relatively open in $\mathcal{N}[\mathbb{K}]$.

- If $U \subseteq \mathbb{K}$ is open and bounded, and $n \in \mathbb{N}$, the set

$$\{\xi : \xi(U) \geq n\} = \{\xi : \xi(U) > n + 1\} \tag{25}$$

is relatively open in $\mathcal{N}[\mathbb{K}]$.

The relative vague topology on $\mathcal{N}[\mathbb{K}]$ has an intuitive description (see Appendix B of [6]): roughly, two particle configurations ξ and ξ' are close to each other if there is a large compact set $C \subseteq \mathbb{K}$ and a small $\varepsilon > 0$ such that the particles of ξ and the particles of ξ' that are in C can be paired in such a way that the paired particles have distance less than ε from each other. (The particles close to the boundary of C are allowed to be paired with those that are outside.) This is similar to Section 11.6 of [2].

If $\xi, \xi' \in \mathcal{N}[\mathbb{K}]$, let us write $\xi \leq \xi'$ if $\xi(B) \leq \xi'(B)$ for every bounded set $B \subseteq \mathbb{K}$. Equivalently, if $\xi = \sum_{a \in Q} n(a)\delta_a$ and $\xi' = \sum_{a \in Q'} n'(a)\delta_a$ are the standard representations of ξ and ξ' (see 1.7), then $\xi \leq \xi'$ if and only if $Q \subseteq Q'$ and $n(a) \leq n'(a)$ for every $a \in Q$. Yet another description is that $\xi \leq \xi'$ if and only if there exists $\xi'' \in \mathcal{N}[\mathbb{K}]$ such that $\xi' = \xi + \xi''$. Clearly, the relation $\leq$ is a partial order on $\mathcal{N}[\mathbb{K}]$.

If $\xi, \xi' \in \mathcal{N}[\mathbb{K}]$ and $\varepsilon > 0$, let us write $\xi \leq_\varepsilon \xi'$ if there exists $\tilde{\xi} \in \mathcal{N}[\mathbb{K} \times \mathbb{K}]$ with marginals $\tilde{\xi}_1 = \tilde{\xi}(\cdot \times \mathbb{K})$ and $\tilde{\xi}_2 = \tilde{\xi}(\mathbb{K} \times \cdot)$ such that

a) $\tilde{\xi}_1 = \xi$ and $\tilde{\xi}_2 \leq \xi'$, and

b) $\tilde{\xi} = \sum_{(a,b)\in\tilde{Q}} \tilde{n}(a,b)\delta_{(a,b)}$ (the standard representation, see 1.7) where $\rho(a,b) < \varepsilon$. (Recall: ρ is the metric on $\mathbb{K}$.)

In words, $\xi \leq_\varepsilon \xi'$ means that there is a matching between particles in ξ and particles in ξ' that covers all the particles in ξ, and such that the matched particles have distance less than ε. Let us call a matching of particles in ξ and ξ' an ε-matching if every two matched particles have distance less than ε.

Let $\xi = \sum_{a\in Q} n(a)\delta_a$ be the standard representation of ξ (see 1.7). It follows from Hall's marriage theorem (e.g., Section 5 of [11]) that $\xi \leq_\varepsilon \xi'$ if and only if $\xi(I) \leq \xi'\left(N_\varepsilon(I)\right)$ for every finite $I \subseteq Q$ (recall: $N_\varepsilon(I)$ is the set of points with distance less than ε from I). The latter condition, in turn, is satisfied if and only if $\xi(B) \leq \xi'\left(N_\varepsilon(B)\right)$ for every compact set $B \subseteq \mathbb{K}$.

Let ξ be a particle configuration, $C \subseteq \mathbb{K}$ a compact set, and $\varepsilon > 0$, and assume that $N_\varepsilon(C)$ is bounded. (The last condition is automatically satisfied if $\mathbb{K} = \mathbb{R}^d$.) Define the cylinder set

$$[\xi]_{C,\varepsilon} \triangleq \left\{\xi' : \xi_C \leq_\varepsilon \xi' \text{ and } \xi'_C \leq_\varepsilon \xi\right\} \ . \tag{26}$$

Note that, if there is an ε-matching of ξ and ξ' that covers the particles of ξ_C, and an ε-matching of ξ and ξ' that covers the particles of ξ'_C, then there is also an ε-matching of ξ and ξ' that covers the particles of both ξ_C and ξ'_C. Therefore, the cylinder $[\xi]_{C,\varepsilon}$ is simply the set of configurations ξ' for which there exists an ε-matching of ξ and ξ' that covers the particles of both ξ_C and ξ'_C.

Each cylinder set is open in the (relative) vague topology.

Argument. We have

$$[\xi]_{C,\varepsilon} = [\xi]^+_{C,\varepsilon} \cap [\xi]^-_{C,\varepsilon} \ , \tag{27}$$

where

$$[\xi]^+_{C,\varepsilon} = \left\{\xi' : \xi_C \leq_\varepsilon \xi'\right\} \ , \tag{28}$$

$$[\xi]^-_{C,\varepsilon} = \left\{\xi' : \xi'_C \leq_\varepsilon \xi\right\} \ . \tag{29}$$

Let $\xi = \sum_{a\in Q} n(a)\delta_a$ be the standard representation of ξ (see 1.7). By Hall's theorem (see above), the set $[\xi]^+_{C,\varepsilon}$ can be written as

$$[\xi]^+_{C,\varepsilon} = \left\{\xi' : \text{for all } I \subseteq Q \cap C,\ \xi'\left(N_\varepsilon(I)\right) \geq \xi(I)\right\} \tag{30}$$

$$= \bigcap_{\substack{I \subseteq Q\cap C \\ \text{finite}}} \left\{\xi' : \xi'\left(N_\varepsilon(I)\right) \geq \xi(I)\right\} \ . \tag{31}$$

Since $N_\varepsilon(I)$ is open, and $Q \cap C$ is finite, we find that $[\xi]^+_{C,\varepsilon}$ is open.

For two particle configurations η and η', let us write $\eta \approx_\varepsilon \eta'$ if $\eta \leq_\varepsilon \eta'$ and $\eta' \leq_\varepsilon \eta$. If $\eta \approx_\varepsilon \eta'$, there is a perfect ε-matching between the particles of η and η' (i.e., an ε-matching that covers the particles of η and η').

The set $[\xi]^-_{C,\varepsilon}$ can be written as

$$[\xi]^-_{C,\varepsilon} = \left\{\xi' : \text{ there exists } \hat{\xi} \le \xi_{N_\varepsilon(C)} \text{ such that } \hat{\xi} \approx_\varepsilon \xi'_C \right\} . \tag{32}$$

The inclusion $\supseteq$ is clear. For $\subseteq$, simply take an ε-matching of ξ and ξ' that covers ξ'_C and remove all the unmatched particles in ξ to obtain $\hat{\xi}$.

The latter, in turn, can be written as

$$[\xi]^-_{C,\varepsilon} = \left\{\xi' : \begin{array}{c} \text{there exists } \hat{\xi} \le \xi_{N_\varepsilon(C)} \text{ and } 0 < \delta < \varepsilon \text{ such that} \\ \xi'\left(\overline{N_\delta(C)}\right) \le \hat{\xi}(\mathbb{K}) \quad \text{and} \quad \hat{\xi} \le_\varepsilon \xi'_{N_\delta(C)} \end{array} \right\} . \tag{33}$$

To see the inclusion $\subseteq$, let $\xi' \in [\xi]^-_{C,\varepsilon}$. Choose $\delta > 0$ small enough so that $\xi'\left(\overline{N_\delta(C)} \setminus C\right) = 0$. (Note that any particle in $\xi'_{\mathbb{K}\setminus C}$ has positive distance from C, and ξ' is locally finite.) Pick an ε-matching of ξ' and ξ that covers ξ'_C. Let $\hat{\xi}$ be the configuration consisting of the matched particles of ξ.

To see the inverse inclusion $\supseteq$, take an ε-matching of $\hat{\xi}$ and $\xi'_{N_\delta(C)}$ that covers $\hat{\xi}$ and such that $\xi'\left(\overline{N_\delta(C)}\right) \le \hat{\xi}(\mathbb{K})$. This is a perfect matching. Removing the particles in $\hat{\xi}$ that are matched with $\xi'_{\overline{N_\delta(C)}\setminus C}$, we obtain a configuration $\hat{\xi}' \le \hat{\xi} \le \xi_{N_\varepsilon(C)}$ that has a perfect ε-matching with ξ'_C.

Finally, exploiting Hall's theorem again, we can rewrite the last expression for $[\xi]^-_{C,\varepsilon}$ as

$$[\xi]^-_{C,\varepsilon} = \bigcup_{\hat{\xi} \le \xi_{N_\varepsilon(C)}} \bigcup_{0<\delta<\varepsilon} \left[\begin{array}{c} \left\{\xi' : \xi'\left(\overline{N_\delta(C)}\right) \le \hat{\xi}(\mathbb{K})\right\} \quad \cap \\ \left\{\xi' : \forall\, I \subseteq \hat{Q}\ , \quad \xi'\left(N_\varepsilon(I) \cap N_\delta(C)\right) \ge \hat{\xi}(I)\right\} \end{array} \right] , \tag{34}$$

where $\hat{Q}$ is the support of $\hat{\xi}$. Note that $\hat{Q}$ is finite. Since $\overline{N_\delta(C)}$ is compact and $N_\varepsilon(I) \cap N_\delta(C)$ is open, we obtain that $[\xi]^-_{C,\varepsilon}$ is open. ⌞

The cylinder sets form a base for the (relative) vague topology on $\mathcal{N}[\mathbb{K}]$.

⌜ *Argument.* Let ξ be a particle configuration. Let $f_1, f_2, \ldots, f_n \in C_c(\mathbb{K})$, and $\varepsilon > 0$. We need to show that the open neighbourhood $\mathcal{U}(\xi, f_1, f_2, \ldots, f_n, \varepsilon) \ni \xi$ (see 2.1) contains a cylinder around ξ.

Let C be a compact neighbourhood of $\bigcup_{i=1}^n \operatorname{supp}(f_i)$ and pick $\alpha > 0$ such that $C \supseteq N_\alpha\left(\bigcup_{i=1}^n \operatorname{supp}(f_i)\right)$. Let $m \triangleq \xi(C)$.

Since f_i are compactly supported, they are uniformly continuous. Pick $0 < \delta < \alpha$ such that for every $a, b \in \mathbb{K}$ with $\rho(a, b) < \delta$, and each i, it holds $|f_i(a) - f_i(b)| < \varepsilon/m$. We claim that

$$[\xi]_{C,\delta} \subseteq \mathcal{U}(\xi, f_1, f_2, \ldots, f_n, \varepsilon) = \bigcap_{i=1}^{n} \mathcal{U}(\xi, f_i, \varepsilon) . \tag{35}$$

Let $\xi' \in [\xi]_{C,\delta}$. Then, there is a δ-matching of ξ and ξ' that covers the particles in $\operatorname{supp}(f_i)$. For each pair $a \sim b$ of matched particles we have $|f_i(a) - f_i(b)| < \varepsilon/m$. There are in total, at most m pairs $a \sim b$ with either $a \in \operatorname{supp}(f_i)$ or $b \in \operatorname{supp}(f_i)$. Therefore, $|\xi(f_i) - \xi'(f_i)| < m \times \varepsilon/m = \varepsilon$. ⌞

If $[\xi]_{C,\varepsilon}$ and $[\xi]_{C',\varepsilon'}$ are cylinders, and $C \subseteq C'$ and $\varepsilon \ge \varepsilon'$, then $[\xi]_{C,\varepsilon} \supseteq [\xi]_{C',\varepsilon'}$. The vague topology on $\mathcal{N}[\mathbb{K}]$, in fact, has a countable base of cylinders.

3.3 Sharp cylinders. Let $\mathbb{K} = \mathbb{R}^d$.

Let ξ be a particle configuration, C a compact set, and $\varepsilon > 0$. Let $\xi = \sum_{a\in Q} n(a)\delta_a$ be the standard representation of ξ (see 1.7). Let us say that the cylinder $[\xi]_{C,\varepsilon}$ is sharp, if

i) $\inf\{\rho(a, b) : a, b \in Q \cap C, a \ne b\} > 2\varepsilon$ (i.e., there exists $\alpha_1 > 2\varepsilon$ such that every two particles that are not on the same position have distance at least α_1 from each other), and

ii) $\inf\{\rho(a, \partial C) : a \in Q\} > \varepsilon$ (i.e., there exists $\alpha_2 > \varepsilon$ such that each $a \in Q$ has distance at least α_2 from the boundary of C).

Every cylinder $[\xi]_{C,\varepsilon}$ around ξ contains a a sharp cylinder $[\xi]_{C',\varepsilon'}$ around ξ.

⌜ *Argument.* Let $D \supseteq C$ be a compact neighbourhood of C. Then, $Q \cap (D \setminus C)$ is finite. Therefore, C and $Q \cap (D \setminus C)$ have positive distance δ from each other. Set $C' \triangleq \overline{N_{\delta/2}(C)}$. Then, each particle of ξ has distance at least $\delta/2$ from $\partial C'$.

Next, let $\gamma \triangleq \inf\{\rho(a,b) : a, b \in Q \cap C', a \neq b\}$. Since ξ has only finitely many particles in C', γ is strictly positive.

⌞ Set $\varepsilon' \triangleq \min\{\delta/3, \gamma/3, \varepsilon\}$.

Therefore, sharp cylinders form a base for the vague topology on $\mathcal{N}[\mathbb{K}]$. Moreover, there is a countable base that consists of sharp cylinders.

3.4 Continuous functions on $\mathcal{N}[\mathbb{K}]$.

4 Probability Measures on Particle Configurations

4.1 Borel σ-algebra on $\mathcal{M}[\mathbb{K}]$. The following σ-algebras on $\mathcal{M}[\mathbb{K}]$ coincide (Lemmas 1.4 and 4.1 in [8]).

$\mathscr{F}_1$ the σ-algebra generated by $\mu \mapsto \mu(B)$ for $B \in \mathscr{E}$.
(Recall: $\mathscr{E}$ denotes the family of bounded measurable subsets of $\mathbb{K}$.)

$\mathscr{F}_2$ the σ-algebra generated by $\mu \mapsto \mu(f)$ for $f \in C_c(\mathbb{K})$.

$\mathscr{F}_3$ the Borel σ-algebra for the vague topology.

Proof.

($\mathscr{F}_2 \subseteq \mathscr{F}_3$) Continuous functions are Borel-measurable.

($\mathscr{F}_3 \subseteq \mathscr{F}_2$) Since the vague topology is second countable (it is separable and metric; see 2.4 and 2.6), every open set is a countable union of finite intersections of sets of the form $\mathcal{U}(\mu, f, \varepsilon) \triangleq \{\nu : |\nu(f) - \mu(f)| < \varepsilon\}$ for $f \in C_c(\mathbb{K})$. Therefore, any vaguely open set is in $\mathscr{F}_2$.

($\mathscr{F}_2 \subseteq \mathscr{F}_1$) If f is a simple function (i.e, it has the form $f = \sum_{i=1}^n \alpha_i 1_{B_i}$ for $B_i \in \mathscr{E}$ and $\alpha_i \geq 0$), then $\mu \mapsto \mu(f)$ is $\mathscr{F}_1$-measurable. If $f \geq 0$, then f is a monotone limit of simple functions, and by the monotone continuity of the measures, we have that $\mu \mapsto \mu(f)$ is a pointwise limit of measurable functions, hence measurable. For arbitrary $f \in C_c(\mathbb{K})$, let $f^+(a) \triangleq \max\{f(a), 0\}$ and $f^- \triangleq \max\{-f(a), 0\}$.

($\mathscr{F}_1 \subseteq \mathscr{F}_2$) If B is compact, there is a decreasing sequence $f_1, f_2, \ldots \in C_c(\mathbb{K})$ such that $f_i \searrow 1_B$ (see 1.4). By monotone continuity of the measures, $\mu(f_i) \searrow \mu(B)$, for each $\mu \in \mathcal{M}[\mathbb{K}]$. Hence, $\mu \mapsto \mu(B)$ is a pointwise limit of $\mathscr{F}_2$-measurable functions.

Now, if $C \subseteq \mathbb{K}$ is a fixed compact set, the family

$$\tilde{\mathscr{B}} \triangleq \{B \subseteq \mathbb{K} \text{ measurable} : \mu \mapsto \mu(B \cap C) \text{ is } \mathscr{F}_2\text{-measurable}\} \tag{36}$$

is a σ-algebra, containing the closed sets, and therefore, coincides with the Borel σ-algebra on $\mathbb{K}$.

To see the latter claim, first note that $\tilde{\mathscr{B}}$ is closed under monotone limits. (That is, if $A_1 \subseteq A_2 \subseteq \cdots$ are in $\tilde{\mathscr{B}}$, so is $\bigcup_i A_i$, and if $A'_1 \supseteq A'_2 \supseteq \cdots$ are in $\tilde{\mathscr{B}}$, so is $\bigcap_i A'_i$.) We show that $\tilde{\mathscr{B}}$ contains an algebra that contains all the closed sets. If so, by the monotone class lemma (e.g., Theorem 4.4.2 of [2] or Lemma 2.35 of [4]), $\tilde{\mathscr{B}}$ contains the Borel σ-algebra.

Let

$$\mathscr{A} \triangleq \{U \cap V : U \subseteq \mathbb{K} \text{ open}, V \subseteq \mathbb{K} \text{ closed}\} . \tag{37}$$

Then, $\mathscr{A}$ is a semi-algebra (i.e., $\varnothing \in \mathscr{A}$, and $E, F \in \mathscr{A}$ implies $E \cap F \in \mathscr{A}$ and $\mathbb{K} \setminus E = \bigcup_{i=1}^n H_i$ for some disjoint $H_1, H_2, \ldots, H_n \in \mathscr{A}$) containing the closed sets. Moreover, $\mathscr{A}$ is included in $\tilde{\mathscr{B}}$. ($U \cap V$ can be written as $V \setminus (V \setminus U)$. Therefore, $\mu(U \cap V \cap C) = \mu(V \cap C) - \mu((V \setminus U) \cap C)$.) The algebra generated by $\mathscr{A}$ has the required property. □

We will denote the Borel σ-algebra on $\mathcal{M}[\mathbb{K}]$ by $\mathscr{F}$. The σ-algebra $\mathscr{F}$ is separable (i.e., generated by a countable family).

[*Argument.* The vague topology is separable and metric (see 2.4 and 2.6), hence has a countable base.

Consequently, there is a countable algebra $\mathscr{A}$ that generates $\mathscr{F}$.

[*Argument.* The algebra generated by a countable generating family is itself countable.

By the monotone class lemma (Theorem 4.4.2 of [2] or Lemma 2.35 of [4]), every two probability measures that agree on $\mathscr{A}$ are equal.

4.2 Restricted σ-algebras on $\mathcal{M}[\mathbb{K}]$. For a measurable $\Lambda \subseteq \mathbb{K}$, we write $\mathscr{F}[\Lambda]$ for the σ-algebra on $\mathcal{M}[\mathbb{K}]$ generated by the mappings $\mu \mapsto \mu(B)$ for bounded measurable $B \subseteq \Lambda$. This is the sub-σ-algebra of events occurring in Λ: it consists of all events $\mathcal{U} \in \mathscr{F}$ such that for each $\mu \in \mathcal{M}[\mathbb{K}]$, whether $\mu \in \mathcal{U}$ depends only on the projection μ_Λ.

If $\mu \in \mathcal{M}[\mathbb{K}]$, the projection μ_Λ can also be seen as an element of $\mathcal{M}[\Lambda]$, the space of particle configurations on Λ. The Borel σ-algebra on $\mathcal{M}[\Lambda]$ induces a σ-algebra on $\mathcal{M}[\mathbb{K}]$ via the mapping $\mu \mapsto \mu_\Lambda$. This σ-algebra coincides with $\mathscr{F}[\Lambda]$.

Argument. If $B \subseteq \Lambda$ and $I \subseteq \mathbb{R}$ are measurable, then

$$\{\mu : \mu_\Lambda(B) \in I\} = \{\mu : \mu(B) \in I\} . \tag{38}$$

Let $\Lambda, \Delta \subseteq \mathbb{K}$ be measurable, and $\Lambda \cap \Delta = \varnothing$. The collection of events of the form $\mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \mathcal{M}[\mathbb{K}]$, where $\mathcal{E}_1 \in \mathscr{F}[\Lambda]$, $\mathcal{E}_2 \in \mathscr{F}[\Delta]$, and $\Lambda \cap \Delta = \varnothing$, constitute a semi-algebra that generates the σ-algebra $\mathscr{F}[\Lambda \cap \Delta]$.

Argument. Let us denote the collection of such sets by $\mathscr{S}$. The fact that $\mathscr{S}$ is a semi-algebra (i.e., $\varnothing \in \mathscr{S}$, and $\mathcal{A}, \mathcal{B} \in \mathscr{S}$ implies $\mathcal{A} \cap \mathcal{B} \in \mathscr{S}$ and $\mathcal{M}[\mathbb{K}] \setminus \mathcal{A} = \bigcup_{i=1}^n \mathcal{C}_i$ for some disjoint $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \in \mathscr{S}$) and is included in $\mathscr{F}[\Lambda \cap \Delta]$ is easy to verify. It remains to verify that $\mathscr{S}$ generates $\mathscr{F}[\Lambda \cap \Delta]$.

For every bounded measurable set $C \subseteq \Lambda \cup \Delta$ and every interval $(a, b) \subseteq \mathbb{R}$ we have

$$\{\mu : \mu(C) \in (a,b)\} = \bigcup_{\substack{x,y,\varepsilon \in \mathbb{Q}, \varepsilon>0 \\ x+y \in (a+2\varepsilon, b-2\varepsilon)}} \left(\{\mu : \mu(C \cap \Lambda) \in (x-\varepsilon, x+\varepsilon)\} \cap \{\mu : \mu(C \cap \Delta) \in (y-\varepsilon, y+\varepsilon)\}\right), \tag{39}$$

which is measurable w.r.t. the σ-algebra generated by $\mathscr{S}$.

In other words, $\mathcal{M}[\Lambda \cup \Delta]$ with σ-algebra $\mathscr{F}[\Lambda \cup \Delta]$ is measure-theoretically isomorphic to $\mathcal{M}[\Lambda] \times \mathcal{M}[\Delta]$ with the product σ-algebra $\mathscr{F}[\Lambda] \otimes \mathscr{F}[\Delta]$ via the mapping $\mu_{\Lambda \cup \Delta} \mapsto (\mu_\Lambda, \mu_\Delta)$ (Lemma 6.1 of [14]). In particular, for every measurable $\Lambda \subseteq \mathbb{K}$, $\mathcal{M}[\Lambda] \times \mathcal{M}[\mathbb{K} \setminus \Lambda]$ is isomorphic to $\mathcal{M}[\mathbb{K}]$.

The intersection $\mathscr{T} \triangleq \bigcap_{\Lambda \in \mathscr{E}} \mathscr{F}[\mathbb{K} \setminus \Lambda]$ is the tail σ-algebra.

4.3 Almost surely continuous projections. The projections $\xi \mapsto \mu_\Lambda$ (for measurable $\Lambda \subseteq \mathbb{K}$) are not continuous. In particular, although $\mathcal{N}[\Lambda] \times \mathcal{N}[\mathbb{K} \setminus \Lambda]$ and $\mathcal{N}[\mathbb{K}]$ are measure-theoretically isomorphic (see 4.2), they are not homeomorphic (taking limit, particles approaching the boundary of Λ may fall in or off Λ). This will cause some trouble when working with specifications and Gibbs measures.

However, the projection $\xi \mapsto \xi_\Lambda$ is continuous at any configuration η that has no particle on the boundary of Λ (i.e., $\eta(\partial\Lambda) = 0$).

Argument. Let $[\eta_\Lambda]_{C,\varepsilon}$ be a cylinder around η_Λ in $\mathcal{N}[\Lambda]$.

Let $\eta = \sum_{a \in Q} n(a)\delta_a$ be the standard representation of η (see 1.7). Let $\delta_0 \triangleq \inf\{\rho(a, \partial(\Lambda \cap C)) : a \in Q\}$ be the minimum distance of the particle of η from the boundary of $\Lambda \cap C$.

Choose $\delta < \min\{\varepsilon, \delta_0\}$. Then, the δ-ball around each particle $a \in Q$ is either completely inside $\Lambda \cap C$ or completely outside $\Lambda \cap C$. Therefore, the projection $\xi \mapsto \xi_\Lambda$ maps the cylinder $[\eta]_{C,\delta}$ into $[\eta_\Lambda]_{C,\varepsilon}$.

Let π be a probability measure on $\mathcal{N}[\mathbb{K}]$, and let $\Lambda \subseteq \mathbb{K}$ be such that $\pi\{\xi : \xi(\partial\Lambda) \neq 0\} = 0$. Then, the projection $\xi \mapsto \xi_\Lambda$ is π-almost surely continuous. For example, if λ is a Radon measure on $\mathbb{K} = \mathbb{R}^d$ that is absolutely continuous w.r.t. the Lebesgue measure, and if $\Lambda \subseteq \mathbb{K}$ is such that $\partial\Lambda$ has Lebesgue measure 0, then the projection $\xi \mapsto \xi_\Lambda$ is almost surely continuous w.r.t. the Poisson measure π^λ.

4.4 How to specify a probability measure on $\mathcal{M}[\mathbb{K}]$. By Ulam's theorem (Theorem 7.1.4 of [2]), every probability measure π on the complete separable metric space $\mathcal{M}[\mathbb{K}]$ is regular; that is,

$$\pi(\mathcal{E}) = \sup\{\pi(\mathcal{C}) : \text{compact } \mathcal{C} \subseteq \mathcal{E}\} \tag{40}$$

for every measurable $\mathcal{E} \subseteq \mathcal{M}[\mathbb{K}]$. In particular, π is uniquely determined by the probabilities it associates to compact events. If $\mathcal{E} \subseteq \mathcal{M}[\mathbb{K}]$ is a measurable set and $\delta > 0$, there exist compact sets $\mathcal{C}_\delta, \mathcal{D}_\delta \subseteq \mathcal{M}[\mathbb{K}]$ with $\mathcal{C}_\delta \subseteq \mathcal{E}$ and $\mathcal{D}_\delta \cap \mathcal{E} = \varnothing$ such that $\pi(\mathcal{C}_\delta \cup \mathcal{D}_\delta) > 1 - \delta$. Since $\mathcal{C}_\delta$ and $\mathcal{D}_\delta$ are disjoint, they have a positive distance from each other, and by Urysohn's lemma, there is a continuous function $\Phi_\delta : \mathcal{M}[\mathbb{K}] \to [0,1]$ such that $\Phi_\delta(\xi) = 1$ for each $\xi \in \mathcal{C}_\delta$, and $\Phi_\delta(\xi) = 0$ for every $\xi \in \mathcal{D}_\delta$. Clearly, $\pi(\Phi_\delta) \to \pi(\mathcal{E})$ as $\delta \to 0$. Therefore, π is also uniquely determined by the expected value it assigns to bounded continuous functions.

The distribution of a random element $\boldsymbol{\mu}$ of $\mathcal{M}[\mathbb{K}]$ can also be specified by either of the following data (Theorem 3.1 of [8]):

- The finite-dimensional joint distributions of $\boldsymbol{\mu}(B)$ for $B \in \mathscr{E}$.
  (Recall: $\mathscr{E}$ denotes the family of bounded subsets of $\mathbb{K}$.)

- The distribution of $\boldsymbol{\mu}(f)$ for each $f \in C_{\mathrm{c}}(\mathbb{K})$.

For each $B \in \mathscr{E}$ and each measurable $I \subseteq \mathbb{R}$, define the event

$$\mathcal{E}_{B,I} \triangleq \{\mu \in \mathcal{M}[\mathbb{K}] : \mu(B) \in I\} \ . \tag{41}$$

Then, the family $\mathscr{S}$ of the sets of the form

$$\mathcal{E}_{B_1,I_1} \cap \mathcal{E}_{B_2,I_2} \cap \cdots \cap \mathcal{E}_{B_n,I_n} \tag{42}$$

is a semi-algebra that generates the Borel σ-algebra on $\mathcal{M}[\mathbb{K}]$ (see 4.1). Therefore, by Carathéodory's extension theorem (e.g., Theorem 3.1.4 of [2] or Theorem 1.14 of [4]), any Borel probability measure on $\mathcal{M}[\mathbb{K}]$ is uniquely determined by the probabilities it assigns to the elements of $\mathscr{S}$. Moreover, every countably additive function $\pi : \mathscr{E} \to [0,\infty)$ with $\pi(\varnothing) = 0$ and $\pi(\mathcal{M}[\mathbb{K}]) = 1$ extends to a (unique) Borel probability measure.

For each $f \in C_{\mathrm{c}}(\mathbb{K})$ and each measurable $I \subseteq \mathbb{R}$, define the event

$$\mathcal{E}_{f,I} \triangleq \{\mu \in \mathcal{M}[\mathbb{K}] : \mu(f) \in I\} \ . \tag{43}$$

Then, the family $\mathscr{S}'$ of the sets of the form

$$\mathcal{E}_{f_1,I_1} \cap \mathcal{E}_{f_2,I_2} \cap \cdots \cap \mathcal{E}_{f_n,I_n} \tag{44}$$

is a semi-algebra that generates the Borel σ-algebra on $\mathcal{M}[\mathbb{K}]$ (see 4.1). Therefore, by Carathéodory's extension theorem, any Borel probability measure on $\mathcal{M}[\mathbb{K}]$ is uniquely determined by the probabilities it assigns the elements of $\mathscr{S}'$. Moreover, every countably additive function $\pi : \mathscr{E} \to [0,\infty)$ with $\pi(\varnothing) = 0$ and $\pi(\mathcal{M}[\mathbb{K}]) = 1$ extends to a (unique) Borel probability measure.

In fact, the probabilities $\pi(\mathcal{E}_{f,I})$ alone are sufficient to uniquely determine the probability measure π.

Argument. (see [8], Theorem A 5.1) Let $f_1, f_2, \ldots, f_n : \mathbb{K} \to \mathbb{R}$ be compactly supported continuous functions. Then, there is a number $0 < L < \infty$ such that $\|f_i\| < L$ for each i. Every probability measure π on $\mathcal{M}[\mathbb{K}]$ induces a probability measure λ on $[-L,L]^n$ via

$$\lambda(I_1 \times I_2 \times \cdots \times I_n) \triangleq \pi\left(\mathcal{E}_{f_1,I_1} \cap \mathcal{E}_{f_2,I_2} \cap \cdots \cap \mathcal{E}_{f_n,I_n}\right) \tag{45}$$

$$= \pi\left\{\mu : (\mu(f_1), \mu(f_2), \ldots, \mu(f_n)) \in I_1 \times I_2 \times \cdots \times I_n\right\} \tag{46}$$

for every measurable $I_1, I_2, \ldots, I_n \subseteq [-L,L]$. This is the joint distribution, with respect to π, of the integrals of $f_1, f_2, \ldots, f_n$. By the regularity of probability measures on $[-L,L]^n$ and using Urysohn's lemma, the measure λ is uniquely determined by the integral $\lambda(g)$ of continuous functions $g : [-L,L]^n \to \mathbb{R}$. Every such continuous function g can be uniformly approximated by linear combinations of functions of the form $(x_1, x_2, \ldots, x_n) \mapsto \mathrm{e}^{-\sum_{i=1}^n \alpha_i x_i}$ for $\alpha_i \in \mathbb{R}$ (using the Stone-Weierstrass theorem). It follows that the measure λ is uniquely determined by the integral of the functions of the form $g(x_1, x_2, \ldots, x_n) \triangleq \mathrm{e}^{-\sum_{i=1}^n \alpha_i x_i}$.

Let $g(x_1, x_2, \ldots, x_n) \triangleq \mathrm{e}^{-\sum_{i=1}^n \alpha_i x_i}$. If μ is a Radon measure on $\mathbb{K}$, we have

$$g\left(\mu(f_1), \mu(f_2), \ldots, \mu(f_n)\right) = \mathrm{e}^{-\sum_{i=1}^n \alpha_i \mu(f_i)} \tag{47}$$

$$= \mathrm{e}^{-\mu\left(\sum_{i=1}^n \alpha_i f_i\right)} \ . \tag{48}$$

Let $f \triangleq \sum_{i=1}^n \alpha_i f_i$. The integral $\pi\left(\mu \mapsto \mathrm{e}^{-\mu(f)}\right)$ is uniquely determined by the probabilities $\pi(\mathcal{E}_{f,I})$ where $I \subseteq \mathbb{R}$ is measurable. Therefore, the probability measure λ, and hence the probabilities $\pi\left(\mathcal{E}_{f_1,I_1} \cap \mathcal{E}_{f_2,I_2} \cap \cdots \cap \mathcal{E}_{f_n,I_n}\right)$, are uniquely determined by the probabilities $\pi(\mathcal{E}_{f,I})$ for f in the linear span of $f_1, f_2, \ldots, f_n$ and measurable $I \subseteq \mathbb{R}$.

4.5 How to specify a probability measure on $\mathcal{N}[\mathbb{K}]$. For every bounded measurable set $B \subseteq \mathbb{K}$ and each non-negative integer k, define the event

$$\mathcal{E}_{B,k} \triangleq \{\mu \in \mathcal{M}[\mathbb{K}] : \mu(B) = k\} \ . \tag{49}$$

Every probability measure π on $\mathcal{N}[\mathbb{K}]$ is uniquely identified by the probabilities it associates to the events of the form

$$\mathcal{E}_{A_1,k_1} \cap \mathcal{E}_{A_2,k_2} \cap \cdots \cap \mathcal{E}_{A_m,k_m} \tag{50}$$

where $A_1, A_2, \ldots, A_m \in \mathscr{E}$ are disjoint, and $k_1, k_2, \ldots, k_m$ are non-negative integers.

⌜

Argument. Recall that π is uniquely determined by the probabilities it associates to the events

$$\mathcal{E}_{B_1,I_1} \cap \mathcal{E}_{B_2,I_2} \cap \cdots \cap \mathcal{E}_{B_n,I_n} \tag{51}$$

for bounded measurable (not necessarily disjoint) B_i and measurable $I_i \subseteq \mathbb{R}$ (see 4.4). The intersection of $\bigcap_{i=1}^n \mathcal{E}_{B_i,I_i}$ and $\mathcal{N}[\mathbb{K}]$ can be written as a countable union of sets of the form $\bigcap_{j=1}^m \mathcal{E}_{A_j,k_j}$, where A_j are disjoint.

Namely, let $A_1, A_2, \ldots, A_m \subseteq \bigcup_{i=1}^n B_i$ be all the non-empty intersections

$$\hat{B}_1 \cap \hat{B}_2 \cap \cdots \cap \hat{B}_n \ , \tag{52}$$

where for each i, either $\hat{B}_i = B_i$ or $\hat{B}_i = \mathbb{K} \setminus B_i$. Set

$$J \triangleq \left\{ (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m : \bigcap_{j=1}^m \mathcal{E}_{A_j,k_j} \subseteq \bigcap_{i=1}^n \mathcal{E}_{B_i,I_i} \right\} \ . \tag{53}$$

Then,

$$\mathcal{N}[\mathbb{K}] \cap \bigcap_{i=1}^n \mathcal{E}_{B_i,I_i} = \bigcup_{(k_1,k_2,\ldots,k_m) \in J} \bigcap_{j=1}^m \mathcal{E}_{A_j,k_j} \ , \tag{54}$$

where the terms of the union on the righthand side are disjoint.

⌞

4.6 Probability measures on $\mathcal{M}[\mathbb{K}]$ are regular. The space $\mathcal{M}[\mathbb{K}]$ is separable and has a complete metric (see Section 2). Therefore, by Ulam's theorem (e.g., Theorem 7.1.4 of [2]), every Borel probability measure on $\mathcal{M}[\mathbb{K}]$ is regular.

5 Space of Probability Measures on Particle Configurations

Let $\mathcal{P}[\mathcal{M}[\mathbb{K}]]$ denote the set of Borel probability measures on $\mathcal{M}[\mathbb{K}]$. The weak topology on $\mathcal{P}[\mathcal{M}[\mathbb{K}]]$ is the weakest topology that makes all the mappings $\pi \mapsto \pi(\Phi)$, for bounded continuous functions $\Phi \in BC(\mathcal{M}[\mathbb{K}])$, continuous. In particular, $\pi_n \xrightarrow{\text{w}} \pi$ (π_n converges weakly to π) if and only if $\pi_n(\Phi) \to \pi(\Phi)$ for every $\Phi \in BC(\mathcal{M}[\mathbb{K}])$. If $\boldsymbol{\mu}_n$ and $\boldsymbol{\mu}$ are random Radon measures with distributions π_n and π, respectively, we say that $\boldsymbol{\mu}_n$ converges in distribution to $\boldsymbol{\mu}$ if $\pi_n \xrightarrow{\text{w}} \pi$.

5.1 Set measurements. The following remain valid if $\mathcal{M}[\mathbb{K}]$ is replaced with any metric space.

- For every open set $\mathcal{U} \subseteq \mathcal{M}[\mathbb{K}]$, the mapping $\pi \mapsto \pi(\mathcal{U})$ is lower semi-continuous (i.e., for every $\alpha > 0$, the set $\{\pi : \pi(\mathcal{U}) > \alpha)\}$ is open).
- For every closed set $\mathcal{V} \subseteq \mathcal{M}[\mathbb{K}]$, the mapping $\pi \mapsto \pi(\mathcal{V})$ is upper semi-continuous (i.e., for every $\alpha > 0$, the set $\{\pi : \pi(\mathcal{V}) < \alpha)\}$ is open).
- For every measurable set $\mathcal{B} \subseteq \mathcal{M}[\mathbb{K}]$, the mapping $\pi \mapsto \pi(\mathcal{B})$ is continuous at each point $\nu \in \mathcal{P}[\mathcal{M}[\mathbb{K}]]$ with $\nu(\partial\mathcal{B}) = 0$.

5.2 Criteria for weak convergence. Let $\pi_1, \pi_2, \ldots$ be Borel probability measures on $\mathcal{M}[\mathbb{K}]$. The following conditions are equivalent (e.g., Theorem II.6.1 of [13] or Theorem 2.1 of [1]).

i) $\pi_t \xrightarrow{\mathrm{w}} \pi$ (π_t weakly converges to π),

ii) $\pi_t(\Phi) \to \pi(\Phi)$ for every bounded uniformly continuous function $\Phi : \mathcal{M}[\mathbb{K}] \to \mathbb{R}$,

iii) $\liminf \pi_t(\mathcal{U}) \geq \pi(\mathcal{U})$ for every open set $\mathcal{U} \subseteq \mathcal{M}[\mathbb{K}]$,

iv) $\limsup \pi_t(\mathcal{V}) \leq \pi(\mathcal{V})$ for every closed set $\mathcal{V} \subseteq \mathcal{M}[\mathbb{K}]$,

v) $\pi_t(\mathcal{B}) \to \pi(\mathcal{B})$ for every measurable set $\mathcal{B} \subseteq \mathcal{M}[\mathbb{K}]$ with $\pi(\partial\mathcal{B}) = 0$.

vi) $\pi_t(\Phi) \to \pi(\Phi)$ for every bounded measurable function $\Phi : \mathcal{M}[\mathbb{K}] \to \mathbb{R}$ that is π-almost surely continuous.

Argument. The standard theorem contains the implications (i)⇒(ii)⇒(iii)⇔(iv)⇒(v)⇒(i).

Condition (vi) clearly implies the weak convergence $\pi_t \xrightarrow{\mathrm{w}} \pi$. The proof of the implication (v)⇒(vi) is between the lines of the proof of (v)⇒(i) as, for example, in [1].

Namely, assume that condition (v) holds. Let $\Phi : \mathcal{M}[\mathbb{K}] \to \mathbb{R}$ is a bounded measurable set and $\mathcal{E} \subseteq \mathcal{M}[\mathbb{K}]$ the set of points at which Φ is continuous. Suppose that $\pi(\mathcal{E}) = 1$. We show that $\pi_t(\Phi) \to \pi(\Phi)$. Since Φ is bounded, without loss of generality, and using the linearity of integration, we can assume that Φ takes its values from the interval $[0, 1]$

Using Fubini-Tonelli's theorem, for every probability measure ν on $\mathcal{M}[\mathbb{K}]$, we can write the expected value of Φ as $\nu(\Phi) = \int_0^1 \nu\{\xi : \Phi(\xi) > y\}\mathrm{d}y$. Let $y \in [0, 1]$. Every point at which Φ is continuous is in the interior of $\{\xi : \Phi(\xi) > y\}$. Therefore, $\partial\{\xi : \Phi(\xi) > y\} \subseteq \mathcal{M}[\mathbb{K}] \setminus \mathcal{E}$, which implies $\pi(\partial\{\xi : \Phi(\xi) > y) \leq \pi(\mathcal{M}[\mathbb{K}] \setminus \mathcal{E}) = 0$. Hence, $\pi_t\{\xi : \Phi(\xi) > y\} \to \pi\{\xi : \Phi(\xi) > y\}$. By the dominated convergence theorem,

$$\pi_t(\Phi) = \int_0^1 \pi_t\{\xi : \Phi(\xi) > y\}\mathrm{d}y \to \int_0^1 \pi\{\xi : \Phi(\xi) > y\}\mathrm{d}y = \pi(\Phi)\ . \tag{55}$$

The above are valid on any metric space. In the particular case of $\mathcal{M}[\mathbb{K}]$, there are other more useful equivalent conditions. For every $f \in C_\mathrm{c}(\mathbb{K})$, let us denote the mapping $\mu \mapsto \mu(f)$ by $\Phi_f : \mathcal{M}[\mathbb{K}] \to \mathbb{R}$. Similarly, for $B_1, B_2, \ldots, B_n \in \mathscr{E}$, we write $\Phi_{B_1,B_2,\ldots,B_n} : \mathcal{M}[\mathbb{K}] \to \mathbb{R}^n$ for the mapping $\mu \mapsto (\mu(B_1), \mu(B_2), \ldots, \mu(B_n))$. If $\boldsymbol{\mu}$ is a random Radon measure with probability distribution π, $\Phi_f\pi \triangleq \pi \circ \Phi_f^{-1}$ denotes the probability distribution of $\boldsymbol{\mu}(f)$. The probability measure $\Phi_{B_1,B_2,\ldots,B_n}\pi$ on $\mathbb{R}^n$ is defined similarly.

Let $\pi_1, \pi_2, \ldots$ be Borel probability measures on $\mathcal{M}[\mathbb{K}]$. Either of the following conditions is equivalent to the weak convergence of π_t to π (Theorem 4.2 of [8]).

vii) $\Phi_f\pi_t \xrightarrow{\mathrm{w}} \Phi_f\pi$ for every $f \in C_\mathrm{c}(\mathbb{K})$,

viii) $\Phi_{B_1,B_2,\ldots,B_n}\pi_t \xrightarrow{\mathrm{w}} \Phi_{B_1,B_2,\ldots,B_n}\pi$ for every $n \in \mathbb{N}$ and $B_1, B_2, \ldots, B_n \in \mathscr{E}$ with

$$\pi\{\mu : \mu(\partial B_1) \neq 0\} = \pi\{\mu : \mu(\partial B_2) \neq 0\} = \cdots = \pi\{\mu : \mu(\partial B_n) \neq 0\} = 0\ . \tag{56}$$

On the space $\mathcal{N}[\mathbb{K}]$ of particle configurations, the latter condition has a simpler version.

ix)

$$\pi_t\{\xi:\xi(A_1)=k_1,\xi(A_2)=k_2,\dots,\xi(A_m)=k_m\} \\ \to \pi\{\xi:\xi(A_1)=k_1,\xi(A_2)=k_2,\dots,\xi(A_m)=k_m\} \tag{57}$$

for every $m\in\mathbb{N}$ and disjoint $A_1,A_2,\dots,A_m\in\mathscr{E}$ with

$$\pi\{\xi:\xi(\partial A_1)\neq 0\}=\pi\{\xi:\xi(\partial A_2)\neq 0\}=\cdots=\pi\{\xi:\xi(\partial A_m)\neq 0\}=0\ , \tag{58}$$

and every $k_1,k_2,\dots,k_m\in\mathbb{N}$.

Argument. The above condition is included in Condition (viii). It is therefore enough to verify that Condition (viii) holds whenever the above condition is satisfied.

For probability measures on $\mathcal{N}[\mathbb{K}]$, the measures $\Phi_{B_1,B_2,\dots,B_n}\pi_t$ and $\Phi_{B_1,B_2,\dots,B_n}\pi$ are supported at $\mathbb{N}^n$. Therefore, $\Phi_{B_1,B_2,\dots,B_n}\pi_t \xrightarrow{\mathrm{w}} \Phi_{B_1,B_2,\dots,B_n}\pi$ if and only if

$$\pi_t\left(\bigcap_{i=1}^n \mathcal{E}_{B_i,I_i}\right)\to\pi_t\left(\bigcap_{i=1}^n \mathcal{E}_{B_i,I_i}\right) \tag{59}$$

for every finite $I_1,I_2,\dots,I_n\subseteq\mathbb{N}$ (see e.g., Theorem 2.2 of [1]; for the definition of $\mathcal{E}_{B_i,I_i}$, see 4.4).

Let $B_1,B_2,\dots,B_n\in\mathscr{E}$, and let $I_1,I_2,\dots,I_n\subseteq\mathbb{N}$ be finite. As in 4.5, let $A_1,A_2,\dots,A_m\subseteq\bigcup_{i=1}^n B_i$ be all the non-empty intersections

$$\hat{B}_1\cap\hat{B}_2\cap\cdots\cap\hat{B}_n\ , \tag{60}$$

where for each i, either $\hat{B}_i=B_i$ or $\hat{B}_i=\mathbb{K}\setminus B_i$. Set

$$J\triangleq\left\{(k_1,k_2,\dots,k_m)\in\mathbb{N}^m:\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\subseteq\bigcap_{i=1}^n\mathcal{E}_{B_i,I_i}\right\}\ . \tag{61}$$

Then,

$$\bigcap_{i=1}^n\mathcal{E}_{B_i,I_i}=\bigcup_{(k_1,k_2,\dots,k_m)\in J}\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\ , \tag{62}$$

where the terms of union are disjoint. Note also that J is a finite set.

Suppose that $\pi_t\left(\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\right)\to\pi\left(\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\right)$ for each $(k_1,k_2,\dots,k_m)\in\mathbb{N}^n$. Then also

$$\pi_t\left(\bigcap_{i=1}^n\mathcal{E}_{B_i,I_i}\right)=\sum_{(k_1,k_2,\dots,k_m)\in J}\pi_t\left(\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\right) \\ \to\sum_{(k_1,k_2,\dots,k_m)\in J}\pi\left(\bigcap_{j=1}^m\mathcal{E}_{A_j,k_j}\right)=\pi\left(\bigcap_{i=1}^n\mathcal{E}_{B_i,I_i}\right)\ . \tag{63}$$

The claim follows from the fact that if $B_1,B_2,\dots,B_n$ are continuity sets of a configuration ξ, so are $A_1,A_2,\dots,A_m$. (Recall from 2.2 that the family of continuity sets of ξ is an algebra.) Therefore,

$$\{\xi:\xi(\partial A_j)\neq 0\}\subseteq\bigcup_{i=1}^n\{\xi:\xi(\partial B_i)\neq 0\} \tag{64}$$

for each j, and $\pi\{\xi:\xi(\partial A_j)\neq 0\}\leq\sum_{i=1}^n\pi\{\xi:\xi(\partial B_i)\neq 0\}$.

5.3 The weak topology on $\mathcal{P}[\mathcal{M}[\mathbb{K}]]$ is separable and has a complete metric.

(Theorems II.6.2 and II.6.5 in [13])

5.4 Criteria for weak compactness. The space $\mathcal{M}[\mathbb{K}]$ is a complete separable metric space. Let $\mathcal{Q} \subseteq \mathcal{P}[\mathcal{M}[\mathbb{K}]]$ be a family of probability measures on $\mathcal{M}[\mathbb{K}]$. Then, Prohorov's theorem (Theorem II.6.7 of [13] or Theorem 11.5.4 of [2]) states that the weak closure $\overline{\mathcal{Q}}$ is weakly compact if and only if for every $\varepsilon > 0$, there exists a compact set $\mathcal{K}_\varepsilon \subseteq \mathcal{M}[\mathbb{K}]$ such that $\pi(\mathcal{K}_\varepsilon) \geq 1-\varepsilon$ for all $\pi \in \mathcal{Q}$. Such a family is said to be (uniformly) tight.

The above condition is valid for any complete separable metric space instead of $\mathcal{N}[\mathbb{K}]$. There is also a condition specific to $\mathcal{N}[\mathbb{K}]$ (Lemma 4.5 of [8]): a sequence $\pi_1, \pi_2, \dots$ of probability measures on $\mathcal{N}[\mathbb{K}]$ has a weakly convergent subsequence if and only if

$$\lim_{t\to\infty} \limsup_{n\to\infty} \pi_n\{\xi : \xi(B) > t\} = 0 \tag{65}$$

for every bounded measurable $B \subseteq \mathbb{K}$.

5.5 $\mathcal{P}[\mathcal{N}[\mathbb{K}]]$ is weakly closed in $\mathcal{P}[\mathcal{M}[\mathbb{K}]]$.

6 Poisson Measures

Let λ be Radon measure on $\mathbb{K}$. A Poisson measure with intensity measure (or mean measure) λ is a Borel probability measure π^λ on the space of particle configurations $\mathcal{N}[\mathbb{K}]$ such that

I. for every $A \in \mathscr{E}$, and every $k \in \mathbb{N}$, we have

$$\pi^\lambda\{\xi : \xi(A) = k\} = \mathrm{e}^{-\lambda(A)} \frac{\lambda(A)^k}{k!} , \tag{66}$$

where 0^0 is interpreted as 1.
(Recall: $\mathscr{E}$ denotes the family of bounded measurable subsets of $\mathbb{K}$.)

II. for every disjoint $A_1, A_2, \dots, A_n \in \mathscr{E}$, and every $k_1, k_2, \dots, k_n \in \mathbb{N}$, it holds

$$\pi^\lambda\{\xi : \xi(A_1) = k_1, \xi(A_2) = k_2, \dots, \xi(A_n) = k_n\} = \prod_{i=1}^{n} \pi^\lambda\{\xi : \xi(A_i) = k_i\} . \tag{67}$$

A Poisson random configuration (a.k.a. a Poisson point process) on $\mathbb{K}$ is a random configuration $\boldsymbol{\xi} : \Omega \to \mathcal{N}[\mathbb{K}]$ defined on a probability space $(\Omega, \mathscr{A}, \mathbf{Pr})$ whose distribution is a Poisson measure (i.e., the measure $\boldsymbol{\xi}\,\mathbf{Pr}$ defined by $(\boldsymbol{\xi}\,\mathbf{Pr})(A) \triangleq \mathbf{Pr}\{\omega : \boldsymbol{\xi}(\omega) \in A\}$ is a Poisson measure).

In fact, condition II alone is essentially sufficient for the measure to be Poisson. Prékopa's theorem (Theorem 4 of [7]) states that any atom-less measure on $\mathcal{N}[\mathbb{K}]$ that has no multiple points and satisfies condition II is Poisson.

6.1 The superposition theorem. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ be independent Poisson random configurations with intensity measures $\lambda_1, \lambda_2, \dots$. If $\lambda \triangleq \sum_{n=1}^\infty \lambda_n$ is a Radon measure, then $\boldsymbol{\xi} \triangleq \sum_{n=1}^\infty \boldsymbol{\xi}_n$ is a Poisson random configuration with intensity measure λ (e.g., Section 2.2 of [9]).

Proof. We first verify that $\boldsymbol{\xi} \triangleq \sum_{n=1}^\infty \boldsymbol{\xi}_n$ is almost surely a particle configuration.

Argument. Let $B \subseteq \mathbb{K}$ be a bounded measurable set. Then, $\boldsymbol{\xi}(B) = \sum_{n=1}^{\infty} \boldsymbol{\xi}_n(B)$ is almost surely finite. Namely, by the monotone continuity of expectation, we have $\mathbf{E}[\boldsymbol{\xi}(B)] = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{E}[\boldsymbol{\xi}_i(B)] = \lambda(B)$. Hence, $\boldsymbol{\xi}(B)$ cannot take the value ∞ on a set that has a positive probability.

Since $\mathbb{K}$ is σ-compact, there is a chain $\Lambda_1 \subseteq \Lambda_2 \subseteq \ldots \subseteq \mathbb{K}$ of bounded open sets with $\bigcup_{l=1}^{\infty} \Lambda_l = \mathbb{K}$ (see 1.1). With probability 1, all the values $\boldsymbol{\xi}(\Lambda_1), \boldsymbol{\xi}(\Lambda_2), \ldots$ are finite. Every bounded set is included in Λ_l for some l. It follows that $\boldsymbol{\xi}$ is almost surely Radon. Since $\mathcal{N}[\mathbb{K}]$ is closed in $\mathcal{M}[\mathbb{K}]$ (see 3.1), $\boldsymbol{\xi}$ is almost surely a particle configuration.

Next, we recall that the sum of finitely many independent Poisson random variables is also a Poisson random variable. Namely, if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are independent Poisson random variables, respectively with intensities $\gamma_1, \gamma_2, \ldots, \gamma_n$, then $\mathbf{x} \triangleq \sum_{i=1}^{n} \mathbf{x}_i$ is a Poisson random variable with intensity $\gamma \triangleq \sum_{i=1}^{n} \gamma_i$.

Argument.

$$\mathbf{Pr}\{\mathbf{x} = l\} = \sum_{\substack{a_1,a_2,\ldots,a_n \geq 0 \\ a_1+a_2+\cdots+a_n=l}} \prod_{i=1}^{n} \mathrm{e}^{-\gamma_i} \frac{\gamma_i^{a_i}}{a_i!} \tag{68}$$

$$= \frac{\mathrm{e}^{-(\gamma_1+\gamma_2+\cdots+\gamma_n)}}{l!} \sum_{\substack{a_1,a_2,\ldots,a_n \geq 0 \\ a_1+a_2+\cdots+a_n=l}} \binom{l}{a_1, a_2, \ldots, a_n} \gamma_1^{a_1} \gamma_2^{a_2} \cdots \gamma_n^{a_n} \tag{69}$$

$$= \frac{\mathrm{e}^{-\gamma}}{l!} (\gamma_1 + \gamma_2 + \cdots + \gamma_n)^l \tag{70}$$

$$= \mathrm{e}^{-\gamma} \frac{\gamma^l}{l!} . \tag{71}$$

In particular, for every bounded measurable $B \subseteq \mathbb{K}$ and each $n > 0$, $\sum_{i=1}^{n} \boldsymbol{\xi}_i(B)$ is Poisson with intensity $\sum_{i=1}^{n} \lambda_i(B)$.

The pointwise monotone limit of Poisson random variables is also a Poisson random variable: if $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \cdots$ is a chain of Poisson random variables with intensities $\gamma_1 \leq \gamma_2 \leq \cdots$, and if $\gamma_n \nearrow \gamma < \infty$, then the pointwise limit $\mathbf{x} \triangleq \lim_{n\to\infty} \mathbf{x}_n$ is Poisson with intensity γ.

Argument. For every $l \in \mathbb{N}$, the events $\{\mathbf{x}_n \leq l\}$ form a decreasing chain with $\{\mathbf{x} \leq l\} = \bigcap_n \{\mathbf{x}_n \leq l\}$. The claim follows from the monotone continuity of probability measures and the continuity of $\sum_{i=0}^{l} \mathrm{e}^{-\gamma} \frac{\gamma^i}{i!}$ in γ.

It follows that for every bounded measurable $B \subseteq \mathbb{K}$, $\boldsymbol{\xi}(B) = \sum_{i=1}^{\infty} \boldsymbol{\xi}_i(B)$ is Poisson with intensity $\lambda(B) = \sum_{i=1}^{\infty} \lambda_i(B)$.

The sum of independent random variables are independent: if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$ are independent random variables in $\mathbb{N}$, so are $\mathbf{x}_1 + \mathbf{y}_1, \mathbf{x}_2 + \mathbf{y}_2, \ldots, \mathbf{x}_m + \mathbf{y}_m$. In particular, if $B_1, B_2, \ldots, B_m \subseteq \mathbb{K}$ are disjoint bounded measurable sets and $n > 0$, the variables $\sum_{i=1}^{n} \boldsymbol{\xi}_i(B_1)$, $\sum_{i=1}^{n} \boldsymbol{\xi}_i(B_2)$, $\ldots$, $\sum_{i=1}^{n} \boldsymbol{\xi}_i(B_m)$ are independent.

Finally, the monotone limit of independent random variables are independent: for every $n \in \mathbb{N}$, let $\mathbf{x}_1^{(n)}, \mathbf{x}_2^{(n)}, \ldots, \mathbf{x}_m^{(n)}$ be independent random variables in $\mathbb{N}$, and suppose that for each $k = 1, 2, \ldots, m$, $\mathbf{x}_k^{(n)} \nearrow \mathbf{x}_k$ as $n \to \infty$. Then, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are independent.

Argument. The events $\{\mathbf{x}_1^{(n)} \leq l_1, \mathbf{x}_2^{(n)} \leq l_2, \ldots, \mathbf{x}_m^{(n)} \leq l_m\}$, for $n = 1, 2, \ldots$, form a decreasing chain with

$$\{\mathbf{x}_1 \leq l_1, \mathbf{x}_2 \leq l_2, \ldots, \mathbf{x}_m \leq l_n\} = \bigcap_i \{\mathbf{x}_1^{(n)} \leq l_1, \mathbf{x}_2^{(n)} \leq l_2, \ldots, \mathbf{x}_m^{(n)} \leq l_m\} . \tag{72}$$

By the monotone continuity of the probability measures, we have

$$\mathbf{Pr}\{\mathbf{x}_1 \le l_1,\, \mathbf{x}_2 \le l_2,\, \ldots,\, \mathbf{x}_m \le l_m\} = \lim_{n\to\infty} \mathbf{Pr}\{\mathbf{x}_1^{(n)} \le l_1,\, \mathbf{x}_2^{(n)} \le l_2,\, \ldots,\, \mathbf{x}_m^{(n)} \le l_m\} \tag{73}$$

$$= \lim_{n\to\infty} \prod_{k=1}^{m} \mathbf{Pr}\{\mathbf{x}_k^{(n)} \le l_k\} \tag{74}$$

$$= \prod_{k=1}^{m} \mathbf{Pr}\{\mathbf{x}_k \le l_k\}\,. \tag{75}$$

⌞

Hence, if $B_1, B_2, \ldots, B_m \subseteq \mathbb{K}$ are disjoint bounded measurable sets, the variables $\boldsymbol{\xi}(B_1), \boldsymbol{\xi}(B_2), \ldots, \boldsymbol{\xi}(B_m)$ are independent. □

6.2 Construction of Poisson measures. A probability measure on $\mathcal{N}[\mathbb{K}]$ is uniquely determined by its values on the sets of the form

$$\{\xi : \xi(A_1) = k_1, \xi(A_2) = k_2, \ldots, \xi(A_n) = k_n\} \tag{76}$$

where $A_1, A_2, \ldots, A_n \in \mathscr{E}$ are disjoint (see 4.5). Therefore, the Poisson measure with intensity measure λ, if exists, is unique.

For the existence, we may use an indirect construction as e.g. in Section 2.5 of [9].

Since $\mathbb{K}$ is σ-compact, there are disjoint bounded measurable sets $K_1, K_2, \ldots \subseteq \mathbb{K}$ such that $\bigcup_{k=1}^{\infty} K_n = \mathbb{K}$. Since λ is Radon, $\lambda(K_n) < \infty$ for each n.

On a suitable probability space $(\Omega, \mathscr{A}, \mathbf{Pr})$, let us construct independent random variables

$$\mathbf{N}_n : \Omega \to \mathbb{N} \qquad (n = 1, 2, \ldots) \tag{77}$$

$$\mathbf{a}_n^i : \Omega \to \mathbb{K} \qquad (n = 1, 2, \ldots,\ i = 1, 2, \ldots) \tag{78}$$

such that, for each n, the following conditions hold.

- The variable $\mathbf{N}_n$ has Poisson distribution with intensity $\lambda(K_n)$. (We define a Poisson distribution with intensity 0 as the distribution concentrated at 0.)
- If $\lambda(K_n) > 0$, for each i, the variable $\mathbf{a}_n^i$ has probability distribution $\lambda_n \triangleq \frac{\lambda(\cdot \cap K_n)}{\lambda(K_n)}$. Otherwise, the distribution of $\mathbf{a}_n^i$ could be arbitrary.

We claim that $\boldsymbol{\xi} \triangleq \sum_{n=1}^{\infty} \sum_{i=1}^{\mathbf{N}_n} \delta_{\mathbf{a}_n^i}$ is a random configuration whose distribution is a Poisson measure with intensity measure λ.

Proof. We first need to verify that the mapping

$$\omega = (N_n, a_n^i)_{n,i} \mapsto \xi_\omega = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \delta_{a_n^i} \tag{79}$$

from the product space $(\mathbb{N} \times \mathbb{K}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathcal{N}[\mathbb{K}]$ is measurable.

Argument. It is enough to verify that for every $B \in \mathscr{E}$ (the family of bounded measurable subsets of $\mathbb{K}$) and every measurable $I \subseteq \mathbb{R}$, the set $\{\omega : \xi_\omega(B) \in I\}$ is measurable (see 4.1).

We have

$$\{\omega : \xi_\omega(B) \in I\} = \bigcup_{r \in I \cap \mathbb{N}} \bigcup_{\substack{r_1, r_2, \ldots \in \mathbb{N} \\ \sum_n r_n = r}} \bigcap_{n=1}^{\infty} \{\omega : \xi_\omega(B \cap K_n) = r_n\} \tag{80}$$

and

$$\{\omega : \xi_\omega(B \cap K_n) = r_n\} = \bigcup_{m \geq r_n} \left\{\omega : N_n = m \text{ and } \begin{pmatrix} \text{among } a_n^1, a_n^2, \ldots, a_n^m, \\ \text{exactly } r_n \text{ are in } B \cap K_n \end{pmatrix}\right\} , \tag{81}$$

which are measurable.

Next, we observe that $\boldsymbol{\xi}_n \triangleq \sum_{i=1}^{\mathbf{N}_n} \delta_{\mathbf{a}_n^i}$ is a Poisson random configuration with intensity measure $\lambda(\cdot \cap K_n)$.

Argument. Let $A_1, A_2, \ldots, A_m \in \mathscr{E}$ be disjoint and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. Then, for each $r \in \mathbb{N}$,

$$\begin{aligned} &\mathbf{Pr}\{\boldsymbol{\xi}_n(A_1) = k_1, \boldsymbol{\xi}_n(A_2) = k_2, \ldots, \boldsymbol{\xi}_n(A_m) = k_m \mid \mathbf{N}_n = r\} \\ &= \mathbf{Pr}\{\boldsymbol{\xi}_n(A_0) = k_0, \boldsymbol{\xi}_n(A_1) = k_1, \ldots, \boldsymbol{\xi}_n(A_m) = k_m \mid \mathbf{N}_n = r\} && (82) \\ &= \binom{r}{k_0, k_1, \ldots, k_m} \lambda_n(A_0)^{k_0} \lambda_n(A_1)^{k_1} \cdots \lambda_n(A_m)^{k_m} , && (83) \end{aligned}$$

where $A_0 \triangleq K_n \setminus \bigcup_{i=1}^m A_i$ and $k_0 \triangleq r - \sum_{i=1}^m k_i$. Hence,

$$\begin{aligned} &\mathbf{Pr}\{\boldsymbol{\xi}_n(A_1) = k_1, \boldsymbol{\xi}_n(A_2) = k_2, \ldots, \boldsymbol{\xi}_n(A_m) = k_m\} \\ &= \sum_{r=0}^{\infty} \mathrm{e}^{-\lambda(K_n)} \frac{\lambda(K_n)^r}{r!} \frac{r!}{k_0!\, k_1! \cdots k_m!} \lambda_n(A_0)^{k_0} \lambda_n(A_1)^{k_1} \cdots \lambda_n(A_m)^{k_m} && (84) \\ &= \prod_{i=1}^{m} \mathrm{e}^{-\lambda(A_i)} \frac{\lambda(A_i)^{k_i}}{k_i!} \sum_{k_0=0}^{\infty} \mathrm{e}^{-\lambda(A_0)} \frac{\lambda(A_0)^{k_0}}{k_0!} && (85) \\ &= \prod_{i=1}^{m} \mathrm{e}^{-\lambda(A_i)} \frac{\lambda(A_i)^{k_i}}{k_i!} . && (86) \end{aligned}$$

The countable sum $\boldsymbol{\xi} = \sum_{n=1}^{\infty} \boldsymbol{\xi}_n$ of Poisson random configurations $\boldsymbol{\xi}_n$ with intensity measures $\lambda(\cdot \cap K_n)$ is a Poisson random measure with intensity measure $\lambda = \sum_{n=1}^{\infty} \lambda(\cdot \cap K_n)$ (see 6.1). □

6.3 Poisson measures are positively correlated. As before (see 3.2), for two configurations $\xi, \xi' \in \mathcal{N}[\mathbb{K}]$ we write $\xi \leq \xi'$ if $\xi(B) \leq \xi'(B)$ for every bounded measurable set $B \subseteq \mathbb{K}$ (i.e., every particle of ξ is also present in ξ'). This is a partial order on $\mathcal{N}[\mathbb{K}]$. An event $\mathcal{E}$ is increasing, if it is upward closed, that is, $\xi' \in \mathcal{E}$ whenever there exists $\xi \leq \xi'$ such that $\xi \in \mathcal{E}$. A probability measure π on $\mathcal{N}[\mathbb{K}]$ is positively correlated if $\pi(\mathcal{A} \cap \mathcal{B}) \geq \pi(\mathcal{A})\pi(\mathcal{B})$ for every two increasing event $\mathcal{A}$ and $\mathcal{B}$.

Positive correlation is closed under weak limits. Therefore, for a Radon measure λ on $\mathbb{K} = \mathbb{R}^d$ that is absolutely continuous with respect to the Lebesgue measure, we can use the positive correlation of the Bernoulli measures to argue that the Poisson measure π^λ is positively correlated.

Is there a better, direct proof that π^λ is positively correlated? What if λ is not absolutely continuous w.r.t. the Lebesgue measure? How about when $\mathbb{K}$ is not $\mathbb{R}^d$?

7 Specifications and Gibbs Measures

To simplify the notations, we shall write $\mathscr{N}$ for $\mathscr{N}[\mathbb{K}]$.

7.1 Multi-species Particle Configurations In this section, we consider the particle configurations in which particles are from a finite set of distinguishable types (or species, or colours). If S is a finite set of symbols, a particle configuration whose each particle is marked with an element of S (its type or colour) is represented by a tuple $\underline{\xi} = (\xi^s)_{s\in S}$, where each ξ^s is an untyped configuration (i.e., an element of $\mathscr{N}$).[1]

The space of S-typed particle configurations is thus denoted by $\mathscr{N}^S$. We endow the space $\mathscr{N}^S$ with the product topology ($\mathscr{N}$ having the vague topology). Recall that $\mathscr{F}$ denotes the Borel σ-algebra on $\mathscr{N}$, and for each measurable $\Lambda \subseteq \mathbb{K}$, $\mathscr{F}[\Lambda] \subseteq \mathscr{F}$ denotes the sub-σ-algebra of events occurring in Λ. We write $\mathscr{F}^S$ for the product σ-algebra on $\mathscr{N}^S$, if $\mathscr{N}$ is given the σ-algebra $\mathscr{F}$. This is the same as the Borel σ-algebra on $\mathscr{N}^S$, because $\mathscr{N}$ is metric and separable (see 2.4 and 2.6). Similarly, $\mathscr{F}^S[\Lambda]$ denotes the product σ-algebra on $\mathscr{N}^S$, if $\mathscr{N}$ is given the σ-algebra $\mathscr{F}[\Lambda]$. Equivalently, $\mathscr{F}^S[\Lambda] \subseteq \mathscr{F}^S$ is the sub-σ-algebra of events occurring in Λ.

7.2 Specifications. As before, we denote by $\mathscr{E}$ the family of bounded measurable subsets of $\mathbb{K}$. Let S be a finite set of symbols. A specification on $\mathscr{N}^S$ is a family $P = [P_\Lambda]_{\Lambda\in\mathscr{E}}$ of proper probability kernels P_Λ from $(\mathscr{N}^S, \mathscr{F}^S[\mathbb{K}\setminus\Lambda])$ to $(\mathscr{N}^S, \mathscr{F}^S)$ that satisfy the consistency condition $P_\Delta P_\Lambda = P_\Delta$ for all $\Lambda, \Delta \in \mathscr{E}$ with $\Lambda \subseteq \Delta$. That is, $P_\Lambda : \mathscr{N}^S \times \mathscr{F}^S \to [0,1]$ (for $\Lambda \in \mathscr{E}$) are such that

i) for each configuration $\underline{\omega} \in \mathscr{N}^S$, $P_\Lambda(\underline{\omega}, \cdot)$ is a probability measure on $(\mathscr{N}^S, \mathscr{F}^S)$,

ii) for each event $\mathcal{E} \in \mathscr{F}^S$, $P_\Lambda(\cdot, \mathcal{E})$ is $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable,

iii) for each $\mathcal{E} \in \mathscr{F}^S$ and $\mathcal{A} \in \mathscr{F}^S[\mathbb{K}\setminus\Lambda]$ we have $P_\Lambda(\cdot, \mathcal{E}\cap\mathcal{A}) = P_\Lambda(\cdot,\mathcal{E})1_{\mathcal{A}}(\cdot)$ (i.e., P_Λ is proper),

iv) for every $\omega \in \mathscr{N}^S$ and $\mathcal{E} \in \mathscr{F}^S$,

$$P_\Delta P_\Lambda(\underline{\omega}, \mathcal{E}) \triangleq P_\Delta(\underline{\omega}, P_\Lambda(\cdot, \mathcal{E})) \tag{87}$$

$$\triangleq \int P_\Delta(\underline{\omega}, \mathrm{d}\underline{\xi}) P_\Lambda(\underline{\xi}, \mathcal{E}) = P_\Delta(\underline{\omega}, \mathcal{E}) \ , \tag{88}$$

whenever $\Lambda \subseteq \Delta$ (i.e., P is consistent).

Note that since $\mathscr{N}^S$ is (as a measurable space) isomorphic to $\mathscr{N}^S[\Lambda] \times \mathscr{N}^S[\mathbb{K}\setminus\Lambda]$ (see 4.2 and 7.1), the properness condition can be expressed as follows:

iii) for each $\underline{\omega} \in \mathscr{N}^S$, $\mathcal{E}_{\text{off}} \in \mathscr{F}^S[\mathbb{K}\setminus\Lambda]$ and $\mathcal{E}_{\text{in}} \in \mathscr{F}^S[\Lambda]$, we have

$$P_\Lambda(\underline{\omega}, \mathcal{E}_{\text{off}} \cap \mathcal{E}_{\text{in}}) = \delta_{\underline{\omega}}(\mathcal{E}_{\text{off}}) \cdot P_\Lambda(\underline{\omega}, \mathcal{E}_{\text{in}}) \ , \tag{89}$$

[1] Another approach would be to consider the S-typed particle configurations as (untyped) particle configurations on $S \times \mathbb{K}$. This would lead to essentially the same space of particle configurations, but a different concept of specification.

where $\delta_{\underline{\omega}}$ denotes the Dirac measure concentrated at $\underline{\omega}$). In particular, $P_\Lambda(\underline{\omega},\cdot)$ is uniquely determined by its restriction to $\mathscr{F}^S[\Lambda]$.

If P is a specification on $\mathcal{N}^S$, a Borel probability measure π is said to be specified by P (or π is a Gibbs measure with specification P) if for every $\Lambda \in \mathscr{E}$ and every $\mathcal{A} \in \mathscr{F}^S$, it holds

$$\pi\left(\mathcal{A} \,\middle|\, \mathscr{F}^S[\mathbb{K}\setminus\Lambda]\right) = P_\Lambda(\cdot,\mathcal{A}) \tag{90}$$

π-almost surely, that is, P_Λ is a regular version of π conditioned to the σ-algebra of events occurring outside Λ. The set of measures specified by P is denoted by $\mathcal{G}(P)$. As in the lattice setting (Remark 1.24 of [5]), we have $\pi \in \mathcal{G}(P)$ if and only if $\pi P_\Lambda = \pi$ for all $\Lambda \in \mathscr{E}$, which holds if and only if $\pi P_\Lambda = \pi$ for all Λ in a cofinal subfamily of $\mathscr{E}$. A family $\mathscr{E}_0 \subseteq \mathscr{E}$ is cofinal, if each $\Lambda \in \mathscr{E}$ is contained in some $\Delta \in \mathscr{E}_0$. For example, the family of bounded open subsets of $\mathbb{K}$ is cofinal, of if $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ are bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$, then $\{\Lambda_n : n = 1,2,\ldots\}$ is cofinal. As usual, we write πP_Λ for the measure defined by $\pi P_\Lambda(\mathcal{A}) \triangleq \pi\left(P_\Lambda(\cdot,\mathcal{A})\right)$.

7.3 Examples.

A. Let $\underline{\omega} \in \mathcal{N}^S$. If for each $\Lambda \in \mathscr{E}$ we define a kernel $P^{\underline{\omega}}_\Lambda$ by $P^{\underline{\omega}}_\Lambda(\underline{\omega}',\cdot) \triangleq \delta_{\underline{\omega}_\Lambda \underline{\omega}'_{\mathbb{K}\setminus\Lambda}}$, we get a trivial specification $P^{\underline{\omega}}$ with $\mathcal{G}(P^{\underline{\omega}}) = \{\delta_{\underline{\omega}}\}$.

B. For each $s \in S$, let λ^s be a Radon measure on $\mathbb{K}$, and write $\underline{\lambda} = (\lambda^s)_{s\in S}$. Let $\pi^{\underline{\lambda}}$ denotes the product, over $s \in S$, of Poisson measures π^{λ^s} on $\mathcal{N}$. (For short, we will call $\pi^{\underline{\lambda}}$ the Poisson measure on $\mathcal{N}^S$ with intensity $\underline{\lambda}$.) For each configuration $\underline{\omega} \in \mathcal{N}^S$, let $\delta_{\underline{\omega}}$ denotes the Dirac measure concentrated at $\underline{\omega}$.

For every bounded measurable $\Lambda \subseteq \mathbb{K}$, we can define a proper probability kernel $P^{\underline{\lambda}}_\Lambda$ by

$$P^{\underline{\lambda}}_\Lambda(\underline{\omega}, \mathcal{E}_{\text{off}} \cap \mathcal{E}_{\text{in}}) \triangleq \delta_{\underline{\omega}}(\mathcal{E}_{\text{off}}) \cdot \pi^{\underline{\lambda}}(\mathcal{E}_{\text{in}}) , \tag{91}$$

for every configuration $\underline{\omega} \in \mathcal{N}^S$, and every two events $\mathcal{E}_{\text{off}} \in \mathscr{F}^S[\mathbb{K}\setminus\Lambda]$ and $\mathcal{E}_{\text{in}} \in \mathscr{F}^S[\Lambda]$.

Argument. Recall, from 4.2, that the family

$$\mathscr{S} \triangleq \left\{\mathcal{E}_{\text{off}} \cap \mathcal{E}_{\text{in}} : \mathcal{E}_{\text{off}} \in \mathscr{F}^S[\mathbb{K}\setminus\Lambda] \text{ and } \mathcal{E}_{\text{in}} \in \mathscr{F}^S[\Lambda]\right\} \tag{92}$$

is a semi-algebra generating $\mathscr{F}^S$. To see that $P^{\underline{\lambda}}_\Lambda(\underline{\omega},\cdot)$ extends to a unique probability measure, we should verify that it is countably additive on $\mathscr{S}$. This goes like the construction of the product measure. Let $\mathcal{E}\cap\mathcal{E}' = \bigcup_{i=1}^\infty \mathcal{E}_i \cap \mathcal{E}'_i$ be a disjoint union of elements of $\mathscr{S}$. Since Λ and $\mathbb{K}\setminus\Lambda$ are disjoint, for every two configurations $\underline{\xi}, \underline{\xi}' \in \mathcal{N}^S$, we have $1_{\mathcal{E}}(\underline{\xi})1_{\mathcal{E}'}(\underline{\xi}') = \sum_{i=1}^\infty 1_{\mathcal{E}_i}(\underline{\xi})1_{\mathcal{E}'_i}(\underline{\xi}')$. Integrating first $\underline{\xi}$ w.r.t. $\delta_{\underline{\omega}}$, and then $\underline{\xi}'$ w.r.t. $\pi^{\underline{\lambda}}$ we obtain that $P^{\underline{\lambda}}_\Lambda(\underline{\omega}, \mathcal{E}\cap\mathcal{E}') = \sum_{i=1}^\infty P^{\underline{\lambda}}_\Lambda(\underline{\omega}, \mathcal{E}_i \cap \mathcal{E}'_i)$.

For every $\mathcal{E}_{\text{off}} \cap \mathcal{E}_{\text{in}} \in \mathscr{S}$, the function $P^{\underline{\lambda}}_\Lambda(\cdot, \mathcal{E}_{\text{off}} \cap \mathcal{E}_{\text{in}}) = \pi^{\underline{\lambda}}(\mathcal{E}_{\text{in}}) \cdot 1_{\mathcal{E}_{\text{off}}}(\cdot)$ is clearly $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable. Let $\mathscr{A}$ be the algebra generated by $\mathscr{S}$. Then, every event in $\mathscr{A}$ is a finite disjoint union of elements of $\mathscr{S}$. Hence, for every $\mathcal{E} \in \mathscr{A}$, $P^{\underline{\lambda}}_\Lambda(\cdot,\mathcal{E})$ is a finite sum of $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable functions, which itself is $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable. Next, let $\mathscr{B}$ be the family of sets $\mathcal{E}$ for which $P^{\underline{\lambda}}_\Lambda(\cdot,\mathcal{E})$ is $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable. This is a monotone class containing the algebra $\mathcal{A}$. Hence it contains the σ-algebra $\mathcal{F}^S$.

Therefore, $P^{\underline{\lambda}}_\Lambda$ is a probability kernel from $(\mathcal{N}^S, \mathscr{F}^S[\mathbb{K}\setminus\Lambda])$ to $(\mathcal{N}^S, \mathscr{F}^S)$. Furthermore, by construction, this probability kernel is proper (see 7.2).

The probability kernels $P^{\underline{\lambda}}_\Lambda$ form a specification $P^{\underline{\lambda}}$, which we refer to as the Poisson specification with intensity measure $\underline{\lambda}$.

Argument. We need to verify that $P^{\underline{\lambda}}$ is consistent.

Let $\Lambda \subseteq \Delta$ and $s \in S$. Let $B_1, B_2, \ldots, B_n \subseteq \Delta$ be disjoint (bounded) measurable sets, $k_1, k_2, \ldots, k_n \in \mathbb{N}$. Then, the event $\{\underline{\xi} : \xi^s(B_i) = k_i \text{ for } i = 1, 2, \ldots, n\}$ can be written as the disjoint union

$$\bigcup_{\substack{l_1,l_2,\ldots,l_n\in\mathbb{N}\\ \forall i:\ l_i\leq k_i}} \{\underline{\xi} : \xi^s(B_i \setminus \Lambda) = l_i \text{ for } 1 \leq i \leq n\} \cap \{\underline{\xi} : \xi^s(B_i \cap \Lambda) = k_i - l_i \text{ for } 1 \leq i \leq n\} . \tag{93}$$

Therefore,

$$\begin{aligned}
&P^{\underline{\lambda}}_{\Delta} P^{\underline{\lambda}}_{\Lambda}\Big(\underline{\omega}, \{\underline{\xi} : \xi^s(B_i) = k_i \text{ for each } i\}\Big) \\
&\quad = \int P_\Delta(\underline{\omega}, \mathrm{d}\underline{\eta}) P_\Lambda\left(\underline{\eta}, \{\underline{\xi} :\ \xi^s(B_i \cap \Lambda) = k_i - \eta^s(B_i \setminus \Lambda) \text{ for each } i\}\right) &(94)\\
&\quad = \sum_{\substack{l_1,l_2,\ldots,l_n\in\mathbb{N}\\ \forall i:\ l_i\leq k_i}} P^{\underline{\lambda}}_{\Delta}\Big(\underline{\omega}, \{\underline{\eta} :\ \eta^s(B_i \setminus \Lambda) = l_i \text{ for each } i\}\Big) &(95)\\
&\qquad\qquad \times \pi^{\underline{\lambda}}\Big(\{\underline{\xi} :\ \xi^s(B_i \cap \Lambda) = k_i - l_i \text{ for each } i\}\Big) \\
&\quad = \sum_{\substack{l_1,l_2,\ldots,l_n\in\mathbb{N}\\ \forall i:\ l_i\leq k_i}} \pi^{\lambda^s}\{\eta^s :\ \eta^s(B_i \setminus \Lambda) = l_i \text{ for each } i\} &(96)\\
&\qquad\qquad \times \pi^{\lambda^s}\{\xi^s :\ \xi^s(B_i \cap \Lambda) = k_i - l_i \text{ for each } i\} \\
&\quad = \pi^{\lambda^s}\{\xi^s : \xi^s(B_i) = k_i \text{ for } i = 1, 2, \ldots, n\} &(97)\\
&\quad = P^{\underline{\lambda}}_{\Delta}\Big(\underline{\omega}, \{\underline{\xi} : \xi^s(B_i) = k_i \text{ for each } i\}\Big) . &(98)
\end{aligned}$$

(The fourth equality is because with respect to the Poisson measure π^{λ^s}, the events occurring on Λ and $\Delta \setminus \Lambda$ are independent.) It follows that the s'th marginals of the measures $P^{\underline{\lambda}}_{\Delta} P^{\underline{\lambda}}_{\Lambda}(\underline{\omega}, \cdot)$ and $P^{\underline{\lambda}}_{\Delta}(\underline{\omega}, \cdot)$ agree on the σ-algebra $\mathscr{F}[\Delta]$ (see 4.4). By the properness of $P^{\underline{\lambda}}_{\Delta}$, the s'th marginals of the measures $P^{\underline{\lambda}}_{\Delta} P^{\underline{\lambda}}_{\Lambda}(\underline{\omega}, \cdot)$ and $P^{\underline{\lambda}}_{\Delta}(\underline{\omega}, \cdot)$ agree also on $\mathscr{F}^S[\mathbb{K} \setminus \Delta]$. Therefore, the s'th marginals of $P^{\underline{\lambda}}_{\Delta} P^{\underline{\lambda}}_{\Lambda}(\underline{\omega}, \cdot)$ and $P^{\underline{\lambda}}_{\Delta}(\underline{\omega}, \cdot)$ agree on $\mathscr{F}$. Finally, the agreement of $P^{\underline{\lambda}}_{\Delta} P^{\underline{\lambda}}_{\Lambda}(\underline{\omega}, \cdot)$ and $P^{\underline{\lambda}}_{\Delta}(\underline{\omega}, \cdot)$ follows from the fact that both are product measures and their corresponding marginals agree.

The Poisson measure $\pi^{\underline{\lambda}}$ is the unique Gibbs measure of $P^{\underline{\lambda}}$ (see Remark 1.25 of [5]).

C. We say that $P = [P_\Lambda]_{\Lambda\in\mathscr{E}}$ is a Markovian specification if there exists $M \in \mathscr{E}$ (the neighbourhood of P) such that for every $\Lambda \in \mathscr{E}$, and each event $\mathcal{A} \in \mathscr{F}[\Lambda]$, $P_\Lambda(\cdot, \mathcal{A})$ is $\mathscr{F}[M(\Lambda) \setminus \Lambda]$-measurable. (Recall: $M(\Lambda) \triangleq \{a + b : a \in \Lambda, b \in M\}$.)

Equivalently, P is Markovian if there exists $W \in \mathscr{E}$ such that for every $\Lambda, \Delta \in \mathscr{E}$ with $W(\Lambda) \cap W(\Delta) = \varnothing$ it holds

$$P_{\Lambda\cup\Delta}(\underline{\omega}, \mathcal{A} \cap \mathcal{B}) = P_\Lambda(\underline{\omega}, \mathcal{A}) \cdot P_\Delta(\underline{\omega}, \mathcal{B}) \tag{99}$$

for every configuration $\underline{\omega}$ and all events $\mathcal{A} \in \mathscr{F}[\Lambda]$ and $\mathcal{B} \in \mathscr{F}[\Delta]$.

D. As in the lattice setup, we would like to have a property like the Feller property or quasi-locality that implies the equivalence of Gibbs measures in the sense of Dobrushin-Lanford-Ruelle and Gibbs measures as thermodynamic limits of the Boltzmann distribution.

Unfortunately, in the continuum setting (e.g., when $\mathbb{K} = \mathbb{R}^d$), the Feller property (or quasi-locality) seems to be too much to ask. For a typical specification $P = [P_\Lambda]_{\Lambda\in\mathscr{E}}$, none of the kernels P_Λ is Feller, simply because for a bounded continuous function $\Phi : \mathscr{N}^S \to \mathbb{R}$ and a configuration $\underline{\omega} \in \mathscr{N}^S$, $P_\Lambda(\omega, \Phi)$ is a function of the projection $\underline{\omega}_{\mathbb{K}\setminus\Lambda}$, and the projection map $\underline{\omega} \to \underline{\omega}_{\mathbb{K}\setminus\Lambda}$ is not continuous; taking a limit, particles may fall in or off Λ and drastically affect the distribution inside Λ.

As a substitute, we introduce the almost Feller property. Let us say that a specification $P = [P_\Lambda]_{\Lambda\in\mathscr{E}}$ is almost Feller if for every bounded measurable set $\Lambda \subseteq \mathbb{K}$ and each bounded continuous $\Phi : \mathcal{N}^S \to \mathbb{R}$, the function $P_\Lambda\Phi = P_\Lambda(\cdot,\Phi)$ is continuous at any point $\underline{\omega} \in \mathcal{N}^S$ for which $\underline{\omega}(\partial\Lambda) = 0$ (i.e., $\underline{\omega}$ has no particle on the boundary of Λ).

Let π be a probability measure. For every bounded measurable set $\Lambda \subseteq \mathbb{K}$, there is a bounded open set $\Delta \supseteq \overline{\Lambda}$ such that a random configuration with distribution π has almost surely no particle on the boundary of Δ; that is, $\pi\{\underline{\xi} : \xi^s(\partial\Delta) > 0 \text{ for some } s \in S\} = 0$.

⌜ *Argument.* Let $\underline{\boldsymbol{\xi}}$ be a random configuration with distribution π, and denote by $|\boldsymbol{\xi}| \triangleq \sum_{s\in S} \boldsymbol{\xi}^s$ the configuration obtained from $\underline{\boldsymbol{\xi}}$ by forgetting the type of the particles.

Pick $r > 0$ such that $\overline{N_r(\Lambda)}$ (i.e., the r-neighbourhood around Λ) is bounded (see 1.1). The random variable $|\boldsymbol{\xi}|\,(N_\varepsilon(\Lambda))$ is increasing in ε. So is its expected value $\mathbf{E}[|\boldsymbol{\xi}|\,(N_\varepsilon(\Lambda))]$. An increasing function on a real interval cannot be discontinuous on more than a countable number of points. Pick an $\varepsilon_c \in (0, r)$ at which the expected value $\mathbf{E}[|\boldsymbol{\xi}|\,(N_\varepsilon(\Lambda))]$ is continuous. By monotone continuity we have

$$\mathbf{E}[|\boldsymbol{\xi}|\,(N_{\varepsilon_c}(\Lambda))] = \lim_{\varepsilon\searrow\varepsilon_c} \mathbf{E}[|\boldsymbol{\xi}|\,(N_\varepsilon(\Lambda))] = \mathbf{E}[\lim_{\varepsilon\searrow\varepsilon_c} |\boldsymbol{\xi}|\,(N_\varepsilon(\Lambda))] = \mathbf{E}[|\boldsymbol{\xi}|\,(\overline{N_{\varepsilon_c}(\Lambda)})] \,. \tag{100}$$

Choose $\Delta \triangleq N_{\varepsilon_c}(\Lambda)$. ⌞

In particular, the collection of bounded measurable sets $\Delta \subseteq \mathbb{K}$ whose boundary contain π-almost surely no particle is cofinal. Moreover, we can choose a cofinal sequence $\Delta_1 \subseteq \Delta_2 \subseteq \cdots$ such that π-almost surely no particle appears on the boundary of any of Δ_k; that is, $\pi\{\underline{\xi} : \xi^s(\partial\Delta_k) > 0 \text{ for some } s \in S \text{ and some } k\} = 0$.

7.4 Construction of Gibbs measures. Let P be an almost Feller specification (see 7.3.D). Let μ be an arbitrary probability measure on $\mathcal{N}^S$. Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ be a chain of bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$ (see 1.1 for the existence). If the weak limit $\pi \triangleq \lim_{n\to\infty} \mu P_{\Lambda_n}$ exists, it is a Gibbs measure for P.

⌜ *Argument.* Let $\Delta \in \mathscr{E}$ be such that π-almost surely no particle appears on the boundary of Δ. Let $\Phi : \mathcal{N}^S \to \mathbb{R}$ be a bounded continuous function. By the almost Feller property, $P_\Delta\Phi$ is π-almost surely continuous. By the weak convergence, we have

$$\pi(P_\Delta\Phi) = \lim_{n\to\infty} (\mu P_{\Lambda_n})(P_\Delta\Phi) \tag{101}$$

(see 5.2). Since $\{\Lambda_n\}_n$ is an open cover of the compact set $\overline{\Delta}$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\Lambda_n \supseteq \Lambda_{n_0} \supseteq \Delta$. Therefore, because of the consistency of P, for every $n \geq n_0$, we have $P_{\Lambda_n}P_\Delta = P_{\Lambda_n}$. Hence,

$$\pi(P_\Delta\Phi) = \lim_{n\to\infty} \mu P_{\Lambda_n} P_\Delta \Phi = \lim_{n\to\infty} \mu P_{\Lambda_n}\Phi = \pi(\Phi) \,. \tag{102}$$

Therefore, $\pi P_\Delta = \pi$. Since the collection of bounded measurable sets Δ whose boundary π-almost surely contain no particle is cofinal, we conclude π is a Gibbs measure for P. ⌞

7.5 Extremal Gibbs measures are tail-trivial and vice versa. (Theorem 7.7 in [5])
Let $P = [P_\Lambda]_{\Lambda\in\mathscr{E}}$ be a specification on $\mathcal{N}^S$, and suppose that $\mathcal{G}(P)$ is non-empty. The set $\mathcal{G}(P)$ is convex, because $\pi \mapsto \pi P_\Lambda$ are affine. If P is almost Feller, then $\mathcal{G}(P)$ is also closed.

⌜ *Argument.* The argument is similar to that of 7.4. Let $\pi_1, \pi_2, \ldots$ be a sequence of Gibbs measures for P, and suppose that π_n converges weakly to a measure π.

Let $\Delta \in \mathscr{E}$ be such that π-almost surely no particle appears on the boundary of Δ. Then, for every bounded continuous function $\Phi : \mathcal{N}^S \to \mathbb{R}$, $P_\Delta\Phi$ is π-almost surely continuous, which implies

$$\pi\Phi = \lim_{n\to\infty} \pi_n\Phi = \lim_{n\to\infty} \pi_n P_\Delta\Phi = \pi P_\Delta\Phi \tag{103}$$

(see 5.2). Since the collection of bounded measurable sets Δ whose boundary π-almost surely contain no particle is cofinal, π must be a Gibbs measure for P. ⌞

An element π of $\mathcal{G}(P)$ is said to be extremal if it cannot be written as a non-trivial convex combination of elements of $\mathcal{G}(P)$.

Let π a Gibbs measure for P and $\mathcal{E}$ a tail event in $\mathcal{N}^S$ with $\pi(\mathcal{E}) > 0$. Then, $\pi(\cdot \mid \mathcal{E})$ is also a Gibbs measure for P.

Argument. The proof is as in the lattice setup. Let $\Lambda \in \mathscr{E}$. Since P_Λ is a proper kernel from $(\mathcal{N}^S, \mathscr{F}^S[\mathbb{K}\setminus\Lambda])$ and $\mathcal{E} \in \mathscr{T}^S \subseteq \mathscr{F}^S[\mathbb{K} \setminus \Lambda]$, for every measurable $\mathcal{A} \subseteq \mathcal{N}^S$ we have

$$(\pi(\cdot \mid \mathcal{E})P_\Lambda)(\mathcal{A}) = \pi(P_\Lambda(\cdot, \mathcal{A}) \mid \mathcal{E}) = \frac{\pi(1_\mathcal{E} P_\Lambda(\cdot, \mathcal{A}))}{\pi(\mathcal{E})} \tag{104}$$

$$= \frac{\pi(P_\Lambda(\cdot, \mathcal{A} \cap \mathcal{E}))}{\pi(\mathcal{E})} = \frac{(\pi P_\Lambda)(\mathcal{A} \cap \mathcal{E})}{\pi(\mathcal{E})} = \frac{\pi(\mathcal{A} \cap \mathcal{E})}{\pi(\mathcal{E})} = \pi(\mathcal{A} \mid \mathcal{E}) . \tag{105}$$

Hence, $\pi(\cdot \mid \mathcal{E})P_\Lambda = \pi(\cdot \mid \mathcal{E})$.

Therefore, if π is an extremal element of $\mathcal{G}(P)$, it is tail-trivial (i.e., it assigns probabilities 0 or 1 to every tail event).

Conversely, if π is tail-trivial Gibbs measure for P, it is extremal in $\mathcal{G}(P)$. More generally, if π and ν are two elements of $\mathcal{G}(P)$ that agree on the tail σ-algebra $\mathscr{T}^S$, then $\pi = \nu$.

Argument. The proof is as in the lattice setup, using the backward martingale convergence theorem (e.g., Theorem 10.6.1 of [2]). Let $\mathcal{A}$ be an event in $\mathscr{F}^S$.

Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ be a chain of bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$. We have

$$\mathscr{F}^S[\mathbb{K} \setminus \Lambda_1] \supseteq \mathscr{F}^S[\mathbb{K} \setminus \Lambda_2] \supseteq \cdots \tag{106}$$

and $\mathscr{T}^S = \bigcap_n \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]$, because every $\Delta \in \mathscr{E}$ is included in Λ_n for some n. Therefore, the sequence $\{\pi(\mathcal{A} \mid \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n])\}_n$ is a reverse martingale, and by the backward martingale convergence theorem

$$\pi(\mathcal{A} \mid \mathscr{T}^S) = \lim_{n\to\infty} \pi(\mathcal{A} \mid \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]) \qquad (\pi\text{-almost surely}) \tag{107}$$

$$= \lim_{n\to\infty} P_{\Lambda_n}(\cdot, \mathcal{A}) \qquad (\pi\text{-almost surely}). \tag{108}$$

Similarly,

$$\nu(\mathcal{A} \mid \mathscr{T}^S) = \lim_{n\to\infty} P_{\Lambda_n}(\cdot, \mathcal{A}) \qquad (\nu\text{-almost surely}). \tag{109}$$

Let $\mathcal{Q} \subseteq \mathcal{N}^S$ be the set of configurations ω for which $\{P_{\Lambda_n}(\omega, \mathcal{A})\}_n$ converges as $n \to \infty$, and define $\Psi : \mathcal{N}^S \to \mathbb{R}$ by

$$\Psi(\omega) \triangleq \begin{cases} \lim_{n\to\infty} P_{\Lambda_n}(\omega, \mathcal{A}) & \text{if } \omega \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases} \tag{110}$$

We have

$$\pi(\mathcal{A}) = \pi\big(\pi(\mathcal{A} \mid \mathscr{T}^S)\big) = \pi(\Psi) , \tag{111}$$

$$\nu(\mathcal{A}) = \nu\big(\nu(\mathcal{A} \mid \mathscr{T}^S)\big) = \nu(\Psi) . \tag{112}$$

But Ψ is $\mathscr{T}^S$-measurable. Therefore, $\pi(\Psi) = \nu(\Psi)$ because π and ν agree on $\mathscr{T}^S$.

Therefore, the extremal Gibbs measures for P are precisely those with respect to which the "macroscopic" events (i.e., the tail events) are deterministic.

As a corollary, every two distinct extremal Gibbs measures $\pi, \nu \in \mathcal{G}(P)$ are mutually singular: there exists a tail event $\mathcal{A} \in \mathscr{T}^S$ such that $\pi(\mathcal{A}) = 1$ and $\nu(\mathcal{N}^S \setminus \mathcal{A}) = 1$.

Argument. Since $\pi \neq \nu$, there exists $\mathcal{A} \in \mathscr{T}^S$ such that $\pi(\mathcal{A}) > \nu(\mathcal{A})$. Since π and ν are tail trivial, we must have $\pi(\mathcal{A}) = 1$ and $\nu(\mathcal{A}) = 0$.

7.6 Extremal Gibbs measures are mixing and vice versa. (See Proposition 7.9 in [5].)
A local event in $\mathcal{N}^S$ is an event $\mathcal{A} \in \mathscr{F}$ that occurs in a bounded measurable region $\Lambda \subseteq \mathbb{K}$, that is, $\mathcal{A} \in \mathscr{F}^S[\Lambda]$ (see 4.2). We say that a measure π on $\mathcal{N}^S$ is mixing (or has short-range correlations) if for every local event $\mathcal{A}$,

$$\lim_{\Lambda \uparrow \mathbb{K}} \sup_{\mathcal{B} \in \mathscr{F}^S[\mathbb{K} \setminus \Lambda]} |\pi(\mathcal{A} \cap \mathcal{B}) - \pi(\mathcal{A})\pi(\mathcal{B})| = 0 \,, \tag{113}$$

where the limit $\lim_{\Lambda \uparrow \mathbb{K}}$ is along the net of bounded measurable subsets of $\mathbb{K}$ with inclusion.

Let $P = [P_\Lambda]_{\Lambda \in \mathscr{E}}$ be a specification on $\mathcal{N}^S$, and suppose that $\mathcal{G}(P)$ is non-empty. Then, every extremal element of $\mathcal{G}(P)$ is mixing.

Argument. Let π be an extremal element of $\mathcal{G}(P)$ and $\mathcal{A}$ a local event. Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ be a chain of bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$. Using the backward martingale convergence theorem we have

$$\pi\big(\mathcal{A} \,|\, \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]\big) \to \pi(\mathcal{A} \,|\, \mathscr{T}^S) \tag{114}$$

π-almost surely (see 7.5). Since $\pi(\mathcal{A} \,|\, \mathscr{T}^S)$ is tail measurable, we know from 7.5 that it is π-almost surely constant. This constant must be $\pi(\mathcal{A})$, because $\pi(\mathcal{A}) = \pi\left(\pi(\mathcal{A} \,|\, \mathscr{T}^S)\right)$. Therefore,

$$\pi\big(\mathcal{A} \,|\, \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]\big) \to \pi(\mathcal{A}) \tag{115}$$

π-almost surely.

Let $\varepsilon > 0$. By Egorov's theorem, there exists a measurable set $\mathcal{E} \subseteq \mathcal{N}^S$ with $\pi(\mathcal{E}) > 1 - \varepsilon/2$, over which the above convergence is uniform. Choose n_ε such that for every $n \geq n_\varepsilon$, we have

$$\Big|\pi(\mathcal{A}) - \pi\big(\mathcal{A} \,\big|\, \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]\big)(\omega)\Big| \leq \varepsilon/2 \tag{116}$$

for all $\omega \in \mathcal{E}$. Therefore, for every event $\mathcal{B} \in \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]$, we obtain, by integrating on $\mathcal{B}$ w.r.t. π, that

$$|\pi(\mathcal{A})\pi(\mathcal{B}) - \pi(\mathcal{A} \cap \mathcal{B})| = \left|\int_{\mathcal{B}} \pi(\mathcal{A}) \mathrm{d}\pi - \int_{\mathcal{B}} \pi\big(\mathcal{A} \,\big|\, \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]\big) \mathrm{d}\pi\right| \tag{117}$$

$$\leq \int_{\mathcal{B} \cap \mathcal{E}} \Big|\pi(\mathcal{A}) - \pi\big(\mathcal{A} \,\big|\, \mathscr{F}^S[\mathbb{K} \setminus \Lambda_n]\big)\Big| \mathrm{d}\pi + \varepsilon/2 \tag{118}$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \,. \tag{119}$$

For every bounded measurable $\Delta \supseteq \Lambda_{n_\varepsilon}$, the same bound holds for every $\mathcal{B} \in \mathscr{F}^S[\mathbb{K} \setminus \Delta]$, which concludes the proof.

Conversely, every mixing element of $\mathcal{G}(P)$ is extremal.

Argument. We show that every mixing $\pi \in \mathcal{G}(P)$ is tail trivial. The extremality of π then follows from 7.5.

Let $\mathcal{B}$ be a tail event. Then, for for every local event $\mathcal{A}$, we have, by the mixing property, that $\pi(\mathcal{A} \cap \mathcal{B}) = \pi(\mathcal{A})\pi(\mathcal{B})$. That is, $\mathcal{A}$ and $\mathcal{B}$ are independent under π. The collection of local events (i.e., $\bigcup_{\Lambda \in \mathscr{E}} \mathscr{F}^S[\Lambda]$) is an algebra that generates the σ-algebra $\mathscr{F}$. It follows from the well-known approximation lemma (approximating the elements of $\mathscr{F}$ by the elements of a generating algebra) that $\mathcal{B}$ is independent of every element of $\mathscr{F}$. In particular, $\mathcal{B}$ is independent of itself, that is, $\pi(\mathcal{B} \cap \mathcal{B}) = \pi(\mathcal{B})\pi(\mathcal{B})$. Hence, either $\pi(\mathcal{B}) = 1$ or $\pi(\mathcal{B}) = 0$.

7.7 Extremal decomposition. Let $P = [P_\Lambda]_{\Lambda \in \mathscr{E}}$ be an specification on $\mathcal{N}^S$. Every Gibbs measure $\pi \in \mathcal{G}(P)$ can be written as a unique convex mixture of extremal elements of $\mathcal{G}(P)$. In other words, $\mathcal{G}(P)$ is a Choquet simplex. This follows from Dynkin's theorem (Theorems 3.1 and 5.1 of [3]; see also Section 7.3 of [5]).

`[argument/explanation to be added.]`

8 Single Species Hard-core Gas

In this section, we assume that $\mathbb{K} = \mathbb{R}^d$. In the hard-core gas model, each particle a is imagined to occupy a volume $W(a)$, and we have the constraint that the volume of distinct particles cannot overlap. We call $W(a)$ the van der Waals volume of a. By the exclusion volume of a, $\tilde{W}(a)$, we mean the set of points whose van der Waals volumes intersects that of a.

8.1 The valid configurations. Let $0 \in W \subseteq \mathbb{K}$ be a bounded measurable set. For $a \in \mathbb{K}$, we write

$$W(a) \triangleq \{a + x : x \in W\} , \tag{120}$$

$$W^{-1}(a) \triangleq \{x : a \in W(x)\} , \tag{121}$$

$$\tilde{W}(a) \triangleq W^{-1}(W(a)) = \{x : W(a) \cap W(x) \neq \varnothing\} . \tag{122}$$

The set of valid configurations is

$$\mathcal{X}_W \triangleq \left\{\xi \in \mathcal{N}[\mathbb{K}] : \xi\left(W^{-1}(a)\right) \leq 1 \text{ for every } a \in \mathbb{K}\right\} , \tag{123}$$

or equivalently,

$$\mathcal{X}_W \triangleq \left\{\xi \in \mathcal{N}[\mathbb{K}] : \xi(\{a\}) \cdot \xi\big(\tilde{W}(a)\big) \leq 1 \text{ for every } a \in \mathbb{K}\right\} , \tag{124}$$

If W is open, the set $\mathcal{X}_W$ is vaguely closed.

⌜ *Argument.* If W is open, so is W^{-1}. We have

$$\mathcal{X}_W = \bigcap_{a \in \mathbb{K}} \left\{\xi : \xi\left(W^{-1}(a)\right) \leq 1\right\} , \tag{125}$$

⌞ which is closed, because $W^{-1}(a)$ are open (see 2.2).

In fact, if W is open, the set $\mathcal{X}_W$ is also compact.

⌜ *Argument.* Since $\mathcal{X}_W$ is closed, it is enough to show that for every bounded set $B \subseteq \mathbb{K}$, the values $\xi(B)$, for $\xi \in \mathcal{X}_W$, are bounded (see 2.7). Since $\overline{B}$ is compact, there is a finite number of points $a_1, a_2, \ldots, a_n \in \mathbb{K}$
⌞ such that $\overline{B} \subseteq \bigcup_{i=1}^{n} W^{-1}(a_i)$. Therefore, for every $\xi \in \mathcal{X}_W$, it holds $\xi(B) \leq \sum_{i=1}^{n} \xi\left(W^{-1}(a)\right) \leq n$.

For a configuration $\xi \in \mathcal{N}[\mathbb{K}]$ and a measurable $\Lambda \subseteq \mathbb{K}$, the projection $\xi_\Lambda \triangleq \xi(\cdot \cap \Lambda)$ can be seen either as an element of $\mathcal{N}[\mathbb{K}]$ or as an element of $\mathcal{N}[\Lambda]$. Conversely, every configuration $\xi \in \mathcal{N}[\Lambda]$ may also be seen as a configuration in $\mathcal{N}[\mathbb{K}]$ by defining $\xi(C) \triangleq \xi(C \cap \Lambda)$ whenever $C \not\subseteq \Lambda$. If Λ and Δ are disjoint measurable subsets of $\mathbb{K}$ and $\xi_\Lambda \in \mathcal{N}[\Lambda]$ and $\xi_\Delta \in \mathcal{N}[\Delta]$, we write $\xi_\Lambda \xi_\Delta \triangleq \xi_\Lambda + \xi_\Delta$ for the configuration that has the particles of both ξ_Λ and ξ_Δ.

For a bounded $\Lambda \subseteq \mathbb{K}$, and a configuration $\omega \in \mathcal{N}[\mathbb{K}]$, the set of valid configurations on Λ with boundary condition ω is defined as

$$\tilde{\mathcal{X}}_W[\Lambda \,|\, \omega] \triangleq \left\{\xi_\Lambda \in \mathcal{N}[\Lambda] : (\xi_\Lambda \omega_{\mathbb{K}\setminus\Lambda})\left(W^{-1}(a)\right) \leq \max\{1, \omega_{\mathbb{K}\setminus\Lambda}\left(W^{-1}(a)\right)\} \text{ for every } a \in W(\Lambda)\right\} \tag{126}$$

$$= \left\{\xi_\Lambda \in \mathcal{N}[\Lambda] : \xi_\Lambda(\{a\}) \cdot (\xi_\Lambda \omega_{\mathbb{K}\setminus\Lambda})\big(\tilde{W}(a)\big) \leq 1 \text{ for every } a \in \Lambda\right\} . \tag{127}$$

This is the set of configurations ξ_Λ on Λ such that the van der Waals volumes of the particles in ξ_Λ do not overlap with each other, or with the van der Waals volumes of the particles in $\omega_{\mathbb{K}\setminus\Lambda}$. If ω is itself a valid configuration (i.e., $\omega \in \mathcal{X}_W$), then we simply have

$$\tilde{\mathcal{X}}_W[\Lambda \,|\, \omega] = \left\{\xi_\Lambda \in \mathcal{N}[\Lambda] : \xi_\Lambda \omega_{\mathbb{K}\setminus\Lambda} \in \mathcal{X}_W\right\} \ , \tag{128}$$

but in principle it might be helpful to have boundary conditions that are not valid.

It will be more convenient to work with the space

$$\mathcal{X}_W[\Lambda \,|\, \omega] \triangleq \left\{\xi \in \mathcal{N}[\mathbb{K}] : \xi_\Lambda \in \tilde{\mathcal{X}}_W[\Lambda \,|\, \omega]\right\} \tag{129}$$

$$= \left\{\xi \in \mathcal{N}[\mathbb{K}] : \xi_\Lambda(\{a\}) \cdot (\xi_\Lambda \omega_{\mathbb{K}\setminus\Lambda})\big(\tilde{W}(a)\big) \leq 1 \text{ for every } a \in \Lambda\right\} \ . \tag{130}$$

of all configurations whose projections on Λ are valid and compatible with ω. This space is isomorphic (as a measurable space) to $\tilde{\mathcal{X}}_W[\Lambda \,|\, \omega] \times \mathcal{N}[\mathbb{K} \setminus \Lambda]$ (see 4.2). Note that $\mathcal{X}_W[\Lambda \,|\, \omega]$ is $\mathscr{F}[\Lambda]$-measurable.

Notation: when W is clear from the context, we may drop the subscript in $\mathcal{X}_W[\Lambda \,|\, \omega]$.

8.2 Continuity of compatibility. The mapping $\eta \mapsto 1_{\mathcal{X}_W[\Lambda \,|\, \omega]}(\eta)$ is continuous at any configuration η satisfying the following two conditions:

i) η has no particle on the boundary of Λ (i.e., $\eta(\partial\Lambda) = 0$), and

ii) for every particle a of $\omega_{\mathbb{K}\setminus\Lambda}\eta_\Lambda$ and every particle b of η_Λ distinct from a, we have $b \notin \partial\tilde{W}(a)$.

⌜ *Argument.* First, suppose that $\eta \in \mathcal{X}_W[\Lambda \,|\, \omega]$. Then, there is a number $\delta > 0$ such that

- for every two distinct particles a and b of η_Λ, $W(a)$ and $W(b)$ have distance at least 2δ,
- for every particle a of $\omega_{\mathbb{K}\setminus\Lambda}$, every particle b of η_Λ has distance at least δ from $\tilde{W}(a)$, and
- every particle of η has distance at least δ from $\partial\Lambda$.

Then, every $\eta' \in [\eta]_{\Lambda,\delta}$ is also in $\mathcal{X}_W[\Lambda \,|\, \omega]$.

Next, suppose that $\eta \notin \mathcal{X}_W[\Lambda \,|\, \omega]$. Then, there is a particles a of $\omega_{\mathbb{K}\setminus\Lambda}\eta_\Lambda$ and a particle b of η_Λ distinct from a, such that $b \in \tilde{W}(a)$, but $b \notin \partial\tilde{W}(a)$. Hence, there is a number $\delta > 0$ such that

- $N_{2\delta}(b) \subseteq \tilde{W}(a)$, and
- $N_\delta(b) \subseteq \mathring{\Lambda}$.

⌞ Every $\eta' \in [\eta]_{\Lambda,\delta}$ is also outside $\mathcal{X}_W[\Lambda \,|\, \omega]$.

Similarly, the mapping $\omega \mapsto 1_{\mathcal{X}_W[\Lambda \,|\, \omega]}(\eta)$ is continuous at any configuration ω satisfying the following two conditions:

i) ω has no particle on the boundary of Λ (i.e., $\omega(\partial\Lambda) = 0$), and

ii) for every particle b of η_Λ and every particle a of $\omega_{\mathbb{K}\setminus\Lambda}$, we have $a \notin \partial\tilde{W}(b)$.

⌜ *Argument.* First, suppose that $\eta \in \mathcal{X}_W[\Lambda \,|\, \omega]$. Then, there is a number $\delta > 0$ such that

- for every particle b of η_Λ, every particle a of $\omega_{\mathbb{K}\setminus\Lambda}$ has distance at least δ from $\tilde{W}(b)$, and
- every particle of ω has distance at least δ from $\partial\Lambda$.

Let $C \triangleq \overline{N_\delta(\tilde{W}(\Lambda))}$. Then, every $\omega' \in [\omega]_{C,\delta}$, we also have $\eta \in \mathcal{X}_W[\Lambda \,|\, \omega']$.

Next, suppose that $\eta \notin \mathcal{X}_W[\Lambda \,|\, \omega]$. If there are distinct particles b and b' of η_Λ such that $W(b) \cap W(b') \neq \varnothing$, then $\eta \notin \mathcal{X}_W[\Lambda \,|\, \omega']$ for every $\omega' \in \mathcal{N}$. Otherwise, there is a particle b of η_Λ and a particle a of $\omega_{\mathbb{K}\setminus\Lambda}\eta_\Lambda$, such that $a \in \tilde{W}(b)$, but $a \notin \partial\tilde{W}(b)$. Hence, there is a number $\delta > 0$ such that $N_\delta(a) \subseteq \tilde{W}(b) \setminus \Lambda$. Again,
⌞ let $C \triangleq \overline{N_\delta(\tilde{W}(\Lambda))}$. Then, for every $\omega' \in [\omega]_{C,\delta}$, we also have $\eta \notin \mathcal{X}_W[\Lambda \,|\, \omega']$.

8.3 The specification. Let λ be a Radon measure on $\mathbb{K}$. Let $P^\lambda = [P^\lambda_\Lambda]_{\Lambda\in\mathscr{E}}$ denote the Poisson specification with intensity measure λ (see 7.3.B). The specification of the single-species hard-core gas with van der Waals volume W is defined by conditioning the Poisson specification to the set of valid configurations. The measure λ plays the role of the fugacity. Typically, λ is uniform, that is, a multiple of the Lebesgue measure.

Namely, for every bounded measurable set $\Lambda \subseteq \mathbb{K}$ and every configuration $\omega \in \mathcal{N}$, let

$$P^{W,\lambda}_\Lambda(\omega,\cdot) \triangleq P^\lambda_\Lambda\left(\omega,\cdot \,|\, \mathcal{X}[\Lambda\,|\,\omega]\right) . \tag{131}$$

where $\mathcal{X}[\Lambda\,|\,\omega]$ is the set of configurations whose projection on Λ is valid and compatible with ω (see 8.1). More explicitly, if π^λ denotes the Poisson measure with intensity measure λ, we have

$$P^{W,\lambda}_\Lambda(\omega,\mathcal{E}_{\text{off}}\cap\mathcal{E}_{\text{in}}) \triangleq \delta_\omega(\mathcal{E}_{\text{off}})\cdot\pi^\lambda\big(\mathcal{E}_{\text{in}}\,\big|\,\mathcal{X}[\Lambda\,|\,\omega]\big) , \tag{132}$$

for every two events $\mathcal{E}_{\text{off}} \in \mathscr{F}[\mathbb{K}\setminus\Lambda]$ and $\mathcal{E}_{\text{in}} \in \mathscr{F}[\Lambda]$. The mapping $P^{W,\lambda}_\Lambda$ is a proper probability kernel from $\mathscr{F}[\mathbb{K}\setminus\Lambda]$ to $\mathscr{F}$.

⌜ *Argument.* First note that $\mathcal{X}[\Lambda\,|\,\omega] \in \mathscr{F}[\Lambda]$. Hence the above two definitions are equivalent.

For every configuration $\omega\in\mathcal{N}$, $P^\lambda_\Lambda(\omega,\cdot\,|\,\mathcal{X}[\Lambda\,|\,\omega])$ is clearly a probability measure. For every event $\mathcal{E}\in\mathscr{F}$, the function

$$P^\lambda_\Lambda(\cdot,\mathcal{E}\,|\,\mathcal{X}[\Lambda\,|\,\omega]) = \frac{P^\lambda_\Lambda(\cdot,\mathcal{E}\cap\mathcal{X}[\Lambda\,|\,\omega])}{P^\lambda_\Lambda(\cdot,\mathcal{X}[\Lambda\,|\,\omega])} \tag{133}$$

⌞ is $\mathscr{F}^S[\mathbb{K}\setminus\Lambda]$-measurable. The properness of $P^{W,\lambda}_\Lambda$ is clear from the construction.

The family $P^{W,\lambda} \triangleq [P^{W,\lambda}_\Lambda]_{\Lambda\in\mathscr{E}}$ is a specification — the hard-core gas specification with van der Waals volume W and fugacity measure λ.

⌜ *Argument.* Let $\Lambda,\Delta\subseteq\mathbb{K}$ be bounded measurable sets with $\Lambda\subseteq\Delta$. To prove the consistency, it is enough to verify that

$$P^{W,\lambda}_\Delta\big(\omega, P^{W,\lambda}_\Lambda(\cdot,\mathcal{E}_1\cap\mathcal{E}_2)\big) = P^{W,\lambda}_\Delta(\omega,\mathcal{E}_1\cap\mathcal{E}_2) , \tag{134}$$

for every configuration $\omega\in\mathcal{N}$ and every two events $\mathcal{E}_1\in\mathscr{F}[\Delta\setminus\Lambda]$ and $\mathcal{E}_2\in\mathscr{F}[\Lambda]$. (Recall that such sets $\mathcal{E}_1\cap\mathcal{E}_2$ form a semi-algebra generating $\mathscr{F}[\Delta]$; see 4.2.)

By the definition of $P^{W,\lambda}_\Delta$, we have

$$P^{W,\lambda}_\Delta\big(\omega, P^{W,\lambda}_\Lambda(\cdot,\mathcal{E}_1\cap\mathcal{E}_2)\big) = \frac{\pi^\lambda\big(1_{\mathcal{X}[\Delta\,|\,\omega]}(\cdot)\cdot P^{W,\lambda}_\Lambda(\cdot,\mathcal{E}_1\cap\mathcal{E}_2)\big)}{\pi^\lambda\big(\mathcal{X}[\Delta\,|\,\omega]\big)} \tag{135}$$

$$= \frac{1}{\pi^\lambda\big(\mathcal{X}[\Delta\,|\,\omega]\big)}\int_{\mathcal{X}[\Delta\,|\,\omega]} P^{W,\lambda}_\Lambda(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\,\pi^\lambda(\mathrm{d}\xi) . \tag{136}$$

Recall that the space $\mathcal{N}[\Delta]$ is isomorphic to the product space $\mathcal{N}[\Delta\setminus\Lambda]\times\mathcal{N}[\Lambda]$ (see 4.2). Since the Poisson measure π^λ induces a product measure on $\mathcal{N}[\Delta\setminus\Lambda]\times\mathcal{N}[\Lambda]$, we can use Fubini-Tonelli's theorem to write

$$\int_{\mathcal{X}[\Delta\,|\,\omega]} P^{W,\lambda}_\Lambda(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\,\pi^\lambda(\mathrm{d}\xi) = \int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]}\Big(\int_{\mathcal{X}[\Lambda\,|\,\xi]} P^{W,\lambda}_\Lambda(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\,\pi^\lambda(\mathrm{d}\eta)\Big)\,\pi^\lambda(\mathrm{d}\xi) \tag{137}$$

$$= \int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]} P^{W,\lambda}_\Lambda(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\cdot\Big(\int_{\mathcal{X}[\Lambda\,|\,\xi]}\pi^\lambda(\mathrm{d}\eta)\Big)\,\pi^\lambda(\mathrm{d}\xi) \tag{138}$$

$$= \int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]} P^{W,\lambda}_\Lambda(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\cdot\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi])\,\pi^\lambda(\mathrm{d}\xi) . \tag{139}$$

By the definition of $P_\Lambda^{W,\lambda}$ and a second application of Fubini-Tonelli's theorem, the last integral can be written as

$$\begin{aligned}
&\int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]} P_\Lambda^{W,\lambda}(\xi,\mathcal{E}_1\cap\mathcal{E}_2)\cdot\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi])\,\pi^\lambda(\mathrm{d}\xi)\\
&\quad=\int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]} 1_{\mathcal{E}_1}(\xi)\cdot\frac{\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi]\cap\mathcal{E}_2)}{\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi])}\cdot\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi])\,\pi^\lambda(\mathrm{d}\xi) && (140)\\
&\quad=\int_{\mathcal{X}[\Delta\setminus\Lambda\,|\,\omega_{\mathbb{K}\setminus\Delta}]} 1_{\mathcal{E}_1}(\xi)\cdot\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi]\cap\mathcal{E}_2)\,\pi^\lambda(\mathrm{d}\xi) && (141)\\
&\quad=\int_{\mathcal{X}[\Delta\,|\,\omega]} 1_{\mathcal{E}_1}(\xi)\cdot 1_{\mathcal{E}_2}(\xi)\,\pi^\lambda(\mathrm{d}\xi) && (142)\\
&\quad=\pi^\lambda\big(\mathcal{X}[\Delta\,|\,\omega]\cap\mathcal{E}_1\cap\mathcal{E}_2\big)\;. && (143)
\end{aligned}$$

Hence, we obtain that

$$P_\Delta^{W,\lambda}\big(\omega,P_\Lambda^{W,\lambda}(\cdot,\mathcal{E}_1\cap\mathcal{E}_2)\big)=\frac{\pi^\lambda\big(\mathcal{X}[\Delta\,|\,\omega]\cap\mathcal{E}_1\cap\mathcal{E}_2\big)}{\pi^\lambda\big(\mathcal{X}[\Delta\,|\,\omega]\big)}=P_\Delta^{W,\lambda}(\omega,\mathcal{E}_1\cap\mathcal{E}_2)\;, \tag{144}$$

concluding the proof.

For every configuration $\xi\in\mathcal{N}$ and every measurable observable $\Phi:\mathcal{N}\to\mathbb{R}$, we have

$$\big(P_\Lambda^{W,\lambda}\Phi\big)(\xi)=P_\Lambda^{W,\lambda}(\xi,\Phi)=\frac{P_\Lambda^\lambda\big(\xi,1_{\mathcal{X}[\Lambda\,|\,\xi]}(\cdot)\,\Phi\big)}{\pi^\lambda(\mathcal{X}[\Lambda\,|\,\xi])}\;. \tag{145}$$

Using Fubini-Tonelli's theorem, the numerator can be rewritten as

$$P_\Lambda^\lambda\big(\xi,1_{\mathcal{X}[\Lambda\,|\,\xi]}(\cdot)\,\Phi\big)=\int 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)\cdot\Phi(\xi_{\mathbb{K}\setminus\Lambda}\eta_\Lambda)\,\pi^\lambda(\mathrm{d}\eta)\;. \tag{146}$$

Argument. Recall that $\mathcal{N}[\mathbb{K}]$ is isomorphic (as a measurable space) to $\mathcal{N}[\Lambda^\complement]\times\mathcal{N}[\Lambda]$ (see 4.2), and that the Poisson measure π^λ induces a product measure $\pi^\lambda_{\Lambda^\complement}\times\pi^\lambda_\Lambda$ on $\mathcal{N}[\Lambda^\complement]\times\mathcal{N}[\Lambda]$.

$$\begin{aligned}
P_\Lambda^\lambda\big(\xi,1_{\mathcal{X}[\Lambda\,|\,\xi]}(\cdot)\,\Phi\big)&=\int 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)\cdot\Phi(\eta)\,(\delta_{\xi_{\Lambda^\complement}}\times\pi^\lambda_\Lambda)(\mathrm{d}\eta) && (147)\\
&=\iint 1_{\tilde{\mathcal{X}}[\Lambda\,|\,\xi]}(\eta_\Lambda)\cdot\Phi(\eta_{\Lambda^\complement}\eta_\Lambda)\,\delta_{\xi_{\Lambda^\complement}}(\mathrm{d}\eta_{\Lambda^\complement})\pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) && (148)\\
&=\int 1_{\tilde{\mathcal{X}}[\Lambda\,|\,\xi]}(\eta_\Lambda)\cdot\Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\,\pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) && (149)\\
&=\iint 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta_{\Lambda^\complement}\eta_\Lambda)\cdot\Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\,\pi^\lambda_{\Lambda^\complement}(\mathrm{d}\eta_{\Lambda^\complement})\pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) && (150)\\
&=\int 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)\cdot\Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\,\pi^\lambda(\mathrm{d}\eta)\;. && (151)
\end{aligned}$$

Therefore, we can write

$$\big(P_\Lambda^{W,\lambda}\Phi\big)(\xi)=\frac{\displaystyle\int 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)\cdot\Phi(\xi_{\mathbb{K}\setminus\Lambda}\eta_\Lambda)\,\pi^\lambda(\mathrm{d}\eta)}{\displaystyle\int 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)\,\pi^\lambda(\mathrm{d}\eta)}\;. \tag{152}$$

8.4 Markov property. The specification $P^{W,\lambda}$ is Markovian (see 7.3.C).

⌜ *Argument.* Let $\Lambda, \Delta \in \mathscr{E}$ be such that $W(\Lambda) \cap W(\Delta) = \varnothing$. Then, for every $\omega \in \mathcal{N}$, $\mathcal{X}[\Lambda \cap \Delta \,|\, \omega] = \mathcal{X}[\Lambda \,|\, \omega] \cap \mathcal{X}[\Delta \,|\, \omega]$. Therefore, for every $\mathcal{E}_\Lambda \in \mathscr{F}[\Lambda]$, $\mathcal{E}_\Delta \in \mathscr{F}[\Delta]$ and $\mathcal{E}_{\text{off}} \in \mathscr{F}[\mathbb{K} \setminus (\Lambda \cup \Delta)]$,

$$P^{W,\lambda}_{\Lambda\cup\Delta}(\omega, \mathcal{E}_{\text{off}} \cap \mathcal{E}_\Lambda \cap \mathcal{E}_\Delta) = 1_{\mathcal{E}_{\text{off}}}(\omega) \cdot \frac{\pi^\lambda\big(\mathcal{X}[\Lambda \cap \Delta \,|\, \omega] \cap \mathcal{E}_\Lambda \cap \mathcal{E}_\Delta\big)}{\pi^\lambda\big(\mathcal{X}[\Lambda \cap \Delta \,|\, \omega]\big)} \tag{153}$$

$$= 1_{\mathcal{E}_{\text{off}}}(\omega) \cdot \frac{\pi^\lambda\big(\mathcal{X}[\Lambda \,|\, \omega] \cap \mathcal{X}[\Delta \,|\, \omega] \cap \mathcal{E}_\Lambda \cap \mathcal{E}_\Delta\big)}{\pi^\lambda\big(\mathcal{X}[\Lambda \,|\, \omega] \cap \mathcal{X}[\Delta \,|\, \omega]\big)} \tag{154}$$

$$= 1_{\mathcal{E}_{\text{off}}}(\omega) \cdot \frac{\pi^\lambda\big(\mathcal{X}[\Lambda \,|\, \omega] \cap \mathcal{E}_\Lambda\big) \cdot \pi^\lambda\big(\mathcal{X}[\Delta \,|\, \omega] \cap \mathcal{E}_\Delta\big)}{\pi^\lambda\big(\mathcal{X}[\Lambda \,|\, \omega]\big) \cdot \pi^\lambda\big(\mathcal{X}[\Delta \,|\, \omega]\big)} \tag{155}$$

$$= P^{W,\lambda}_\Lambda(\omega, \mathcal{E}_{\text{off}} \cap \mathcal{E}_\Lambda) \cdot P^{W,\lambda}_\Delta(\omega, \mathcal{E}_{\text{off}} \cap \mathcal{E}_\Delta)\ . \tag{156}$$

⌞

8.5 Almost Feller property. If the exclusion volume $\tilde{W} = W^{-1}(W)$ satisfies $\lambda(\partial \tilde{W}(a)) = 0$ for every $a \in \mathbb{K}$, then the specification $P^{W,\lambda}$ is also almost Feller (see 7.3.D).

Proof. Let $\Lambda \in \mathscr{E}$ and $\omega \in \mathcal{N}$ be such that ω has no particle on the boundary of Λ. Let $\Phi : \mathcal{N} \to \mathbb{R}$ be a bounded continuous function. We have to show that $P^{W,\lambda}_\Lambda \Phi$ is continuous at ω.

For every $\xi \in \mathcal{N}$ we have

$$\big(P^{W,\lambda}_\Lambda \Phi\big)(\xi) = P^{W,\lambda}_\Lambda(\xi, \Phi) = \frac{P^\lambda_\Lambda\big(\xi, 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\cdot)\, \Phi\big)}{\pi^\lambda(\mathcal{X}[\Lambda \,|\, \xi])}\ . \tag{157}$$

Using Fubini-Tonelli's theorem, the numerator can be written as

$$P^\lambda_\Lambda\big(\xi, 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\cdot)\, \Phi\big) = \int 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta) \cdot \Phi(\xi_{\mathbb{K}\setminus\Lambda}\eta_\Lambda)\, \pi^\lambda(\mathrm{d}\eta)\ . \tag{158}$$

⌜ *Argument.* Recall that $\mathcal{N}[\mathbb{K}]$ is isomorphic (as a measurable space) to $\mathcal{N}[\Lambda^{\complement}] \times \mathcal{N}[\Lambda]$ (see 4.2), and that the Poisson measure π^λ induces a product measure $\pi^\lambda_{\Lambda^\complement} \times \pi^\lambda_\Lambda$ on $\mathcal{N}[\Lambda^\complement] \times \mathcal{N}[\Lambda]$.

$$P^\lambda_\Lambda\big(\xi, 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\cdot)\, \Phi\big) = \int 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta) \cdot \Phi(\eta)\, (\delta_{\xi_{\Lambda^\complement}} \times \pi^\lambda_\Lambda)(\mathrm{d}\eta) \tag{159}$$

$$= \iint 1_{\tilde{\mathcal{X}}[\Lambda \,|\, \xi]}(\eta_\Lambda) \cdot \Phi(\eta_{\Lambda^\complement}\eta_\Lambda)\, \delta_{\xi_{\Lambda^\complement}}(\mathrm{d}\eta_{\Lambda^\complement}) \pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) \tag{160}$$

$$= \int 1_{\tilde{\mathcal{X}}[\Lambda \,|\, \xi]}(\eta_\Lambda) \cdot \Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\, \pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) \tag{161}$$

$$= \iint 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta_{\Lambda^\complement}\eta_\Lambda) \cdot \Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\, \pi^\lambda_{\Lambda^\complement}(\mathrm{d}\eta_{\Lambda^\complement}) \pi^\lambda_\Lambda(\mathrm{d}\eta_\Lambda) \tag{162}$$

$$= \int 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta) \cdot \Phi(\xi_{\Lambda^\complement}\eta_\Lambda)\, \pi^\lambda(\mathrm{d}\eta)\ . \tag{163}$$

⌞

Therefore, we can write

$$\big(P^{W,\lambda}_\Lambda \Phi\big)(\xi) = \frac{\displaystyle\int 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta) \cdot \Phi(\xi_{\mathbb{K}\setminus\Lambda}\eta_\Lambda)\, \pi^\lambda(\mathrm{d}\eta)}{\displaystyle\int 1_{\mathcal{X}[\Lambda \,|\, \xi]}(\eta)\, \pi^\lambda(\mathrm{d}\eta)}\ . \tag{164}$$

Recall that the projection $\xi \mapsto \xi_\Lambda$ is continuous at ω, because ω has no particle on the boundary of Λ (see 4.3). For every $\eta \in \mathcal{N}$, the concatenation $\xi_\Lambda \mapsto \xi_\Lambda\eta_\Lambda$ is also clearly continuous. Therefore, $\xi \mapsto \Phi(\xi_{\mathbb{K}\setminus\Lambda}\eta_\Lambda)$ is continuous at ω. Below, we shall verify that $\xi \mapsto 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)$ is also continuous at ω for π^λ-almost every η. If so, it follows, using the dominated convergence theorem, that for every sequence $\xi^1, \xi^2, \ldots$ converging to ω, it holds $\big(P_\Lambda^{W,\lambda}\Phi\big)(\xi^n) \to \big(P_\Lambda^{W,\lambda}\Phi\big)(\omega)$ as $n \to \infty$. That is, $P_\Lambda^{W,\lambda}\Phi$ is continuous at ω.

We now verify that for π^λ-almost every η, the mapping $\xi \mapsto 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)$ is continuous at ω. In fact, $\xi \mapsto 1_{\mathcal{X}[\Lambda\,|\,\xi]}(\eta)$ is continuous at ω if for every particle a of $\omega_{\mathbb{K}\setminus\Lambda}$, the configuration η_Λ has no particle on the boundary of $\tilde{W}(a)$.

⌜ *Argument.* First, suppose that $\eta \in \mathcal{X}[\Lambda\,|\,\omega]$. Then, there is a number $\delta > 0$ such that for every particle a of $\omega_{\mathbb{K}\setminus\Lambda}$ and every particle b of η_Λ, $\tilde{W}(a)$ and b have distance at least δ. Pick a compact set $C \supseteq N_\delta(\tilde{W}(\Lambda))$. Then, for every configuration $\xi \in [\omega]_{C,\delta}$, we have $\eta \in \mathcal{X}[\Lambda\,|\,\xi]$. That is because for every particle a' of $\xi_{\mathbb{K}\setminus\Lambda}$ with $\tilde{W}(a') \cap \Lambda \neq \varnothing$, there is a particle a of $\omega_{\mathbb{K}\setminus\Lambda}$ that has has distance less than δ from a', and every particle of η_Λ has distance at least δ from $\tilde{W}(a)$.

Next, suppose that $\eta \notin \mathcal{X}[\Lambda\,|\,\omega]$. Then, there is a particle a of $\omega_{\mathbb{K}\setminus\Lambda}$ and every particle b of η_Λ such that b is in the interior of $\tilde{W}(a)$. Therefore, there is a number $\delta > 0$ such that for every point a' that has distance less than δ from a, we have $b \in \tilde{W}(a')$. Picking again a compact set $C \supseteq N_\delta(\tilde{W}(\Lambda))$, for every $\xi \in [\omega]_{C,\delta}$ we have $\eta \notin \mathcal{X}[\Lambda\,|\,\xi]$. ⌞

Under the hypothesis $\lambda(\partial\tilde{W}(a)) = 0$, the set of configurations η satisfying the above condition has probability 1 w.r.t. the Poisson measure π^λ.

⌜ *Argument.* We use the random variables used in the construction of the Poisson measure (see 6.2). Namely, let

$$\mathbf{N} : \Omega \to \mathbb{N} \tag{165}$$

$$\mathbf{a}^i : \Omega \to \mathbb{K} \qquad (i = 1, 2, \ldots) \tag{166}$$

be independent random variables on a suitable probability space $(\Omega, \mathscr{A}, \mathbf{Pr})$, where $\mathbf{N}$ has Poisson distribution with intensity $\lambda(\Lambda)$ and each $\mathbf{a}^i$ has distribution $\tilde{\lambda} \triangleq \frac{\lambda(\cdot\cap\Lambda)}{\lambda(\Lambda)}$. (If $\lambda(\Lambda) = 0$, the distribution of $\mathbf{a}^i$ could be chosen arbitrarily.) The random configuration $\boldsymbol{\eta}_\Lambda \triangleq \sum_{i=1}^{\mathbf{N}} \delta_{\mathbf{a}^i}$ has distribution π_Λ^λ. Let c be a particle of $\omega_{\mathbb{K}\setminus\Lambda}$. For every $n \in \mathbb{N}$,

$$\mathbf{Pr}\{\boldsymbol{\eta}_\Lambda(\partial\tilde{W}(a)) > 0\,|\,\mathbf{N} = n\} \leq \sum_{i=1}^{n} \mathbf{Pr}\{\mathbf{a}^i \in \partial\tilde{W}(c)\} = 0\,. \tag{167}$$

It follows that with probability 1, $\boldsymbol{\eta}_\Lambda$ has no particle that is on the boundary of the exclusion volume of a particle of $\omega_{\mathbb{K}\setminus\Lambda}$. ⌞

□

8.6 Existence via compactness. Let $W \subseteq \mathbb{K}$ be a bounded open set with $0 \in W$. Then, the set $\mathcal{X}_W$ of valid configurations of hard-core particles with van der Waals volume W is compact (see 8.1). Let λ be a Radon measure on $\mathbb{K}$. The hard-core specification $P^{W,\lambda}$ has at least one Gibbs measure.

⌜ *Argument.* The set of probability measures on $\mathcal{N}[\mathbb{K}]$ that are supported at $\mathcal{X}_W$ is compact. This follows, for example, from Prohorov's theorem (see 5.4).

Let ω be an arbitrary element of $\mathcal{X}_W$. Then, for every bounded $\Lambda \subseteq \mathbb{K}$, $P_\Lambda^{W,\lambda}(\omega,\cdot)$ is a probability measure supported at $\mathcal{X}_W$. Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ be a chain of bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$. Then, the sequence $\{P_{\Lambda_n}^{W,\lambda}(\omega,\cdot)\}_n$ has a convergent subsequence. Since $P^{W,\lambda}$ is an almost Feller specification (see 8.3), the limit of such a subsequence is a Gibbs measure for P (see 7.4). ⌞

8.7 Existence via domination by Poisson. The existence of hard-core Gibbs measures for arbitrary (bounded measurable) van der Waals volume $W \subseteq \mathbb{K}$ also follows from the fact that the hard-core distributions are dominated by the Poisson measures.

As before (see 3.2 and 6.3) let us order $\mathcal{N}[\mathbb{K}]$ by writing $\xi \leq \xi'$ if every particle in ξ is present in ξ'. This induces a partial ordering $\preceq$ on the space of probability measures $\mathcal{P}[\mathcal{N}[\mathbb{K}]]$ (the domination ordering): $\pi \preceq \pi'$ if $\pi(\mathcal{E}) \leq \pi'(\mathcal{E})$ for every increasing event $\mathcal{E} \subseteq \mathcal{N}[\mathbb{K}]$. A probability measure π is positively correlated if and only if for every decreasing event $\mathcal{E}$ with $\pi(\mathcal{E}) > 0$ it holds $\pi(\cdot \,|\, \mathcal{E}) \preceq \pi$.

Let λ be a Radon measure on $\mathbb{K}$ that is absolutely continuous with respect to the Lebesgue measure. Recall that the Poisson measure π^λ is positively correlated (see 6.3).

Let $W \subseteq \mathbb{K}$ be a bounded measurable set with $0 \in W$. Clearly, for every bounded measurable $\Lambda \subseteq \mathbb{K}$, the set $\mathcal{X}_W[\Lambda \,|\, \omega]$ of configurations that are valid (for the hard-core model) in Λ and compatible with the boundary condition ω is decreasing. (Removing a particle from a valid configuration does not make it invalid.) Therefore, $\pi^\lambda(\cdot \,|\, \mathcal{X}_W[\Lambda \,|\, \omega])$ is dominated by π^λ.

Let $P^\lambda = [P^\lambda_\Lambda]_{\Lambda \in \mathscr{E}}$ denote the Poisson specification (see 7.3), and $P^{W,\lambda} = [P^{W,\lambda}_\Lambda]_{\Lambda \in \mathscr{E}}$ the hard-core specification (see 8.3). Then, for every configuration ω and every increasing event $\mathcal{A} \in \mathscr{F}[\Lambda]$, it holds $P^{W,\lambda}_\Lambda(\omega, \mathcal{A}) \leq P^\lambda_\Lambda(\omega, \mathcal{A})$.

Let ω be an arbitrary configuration. Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ be a chain of bounded open sets with $\bigcup_n \Lambda_n = \mathbb{K}$. Since the sequence $\{P^\lambda_{\Lambda_n}(\omega, \cdot)\}_n$ is convergent (the limit is the Poisson measure π^λ), we have (see 5.4)

$$\lim_{t\to\infty} \limsup_{n\to\infty} P^\lambda_{\Lambda_n}(\omega, \{\xi : \xi(B) > t\}) = 0 \tag{168}$$

for every bounded measurable $B \subseteq \mathbb{K}$. (This can also be seen by calculation.) Since $\{\xi : \xi(B) > t\}$ is an increasing event, and is in $\mathscr{F}[\Lambda]$ for all $\Lambda \supseteq B$, we have

$$P^{W,\lambda}_\Lambda(\omega, \{\xi : \xi(B) > t\}) \leq P^\lambda_\Lambda(\omega, \{\xi : \xi(B) > t\}) \,. \tag{169}$$

for all $\Lambda \supseteq B$. Therefore, for every bounded measurable $B \subseteq \mathbb{K}$, it also holds

$$\lim_{t\to\infty} \limsup_{n\to\infty} P^{W,\lambda}_{\Lambda_n}(\omega, \{\xi : \xi(B) > t\}) = 0 \,. \tag{170}$$

This implies that the sequence $\{P^{W,\lambda}_{\Lambda_n}(\omega, \cdot)\}_n$ has a convergent subsequence (see 5.4). The limit of such subsequence is a Gibbs measure for $P^{W,\lambda}$ (see 7.4).

A Appendix

A.1 Stone-Weierstrass theorem on metric spaces. (Problem 44A of [16], or [12])
Let $\mathcal{X}$ be a metric space. Let $BC(\mathcal{X})$ denote the set of bounded continuous functions $\varphi : \mathcal{X} \to \mathbb{R}$. Let $F \subseteq BC(\mathcal{X})$ be a subalgebra (i.e., a linear subspace that is also closed under multiplication). Then, F coincides with $BC(\mathcal{X})$ if and only if

i) F is closed (w.r.t. the topology of the uniform norm),

ii) F contains the constant functions, and

iii) F separates closed sets in $\mathcal{X}$ (i.e., for every two disjoint closed sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, there is $\varphi \in F$ with $\overline{\varphi(\mathcal{A})} \cap \overline{\varphi(\mathcal{B})} = \varnothing$).

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B List of Symbols

$\mathbb{K}$	The space where particles live. A locally compact complete separable metric space. From some point on it is assumed to be $\mathbb{R}^d$.
$\mathscr{E}$	The family of bounded measurable subsets of $\mathbb{K}$.
ξ	Typical particle configuration on $\mathbb{K}$.
$\boldsymbol{\xi}$	Typical random particle configuration on $\mathbb{K}$.
$\mathcal{N} = \mathcal{N}[\mathbb{K}]$	The space of particle configurations on $\mathbb{K}$.
$\mathcal{N}[\Lambda]$	The space of particle configurations supported at Λ.
μ	Typical Radon measure on $\mathbb{K}$.
$\mathcal{M} = \mathcal{M}[\mathbb{K}]$	The space of Radon measures on $\mathbb{K}$.
$\mathscr{F}$	The Borel σ-algebra on $\mathcal{M}[\mathbb{K}]$ or $\mathcal{N}[\mathbb{K}]$.
$\mathscr{F}[\Lambda]$	The sub-σ-algebra of events occurring at Λ.
$\mathcal{P}[\mathcal{M}[\mathbb{K}]]$	The space of Borel probability measures on $\mathcal{M}[\mathbb{K}]$.
$\mathcal{P}[\mathcal{N}[\mathbb{K}]]$	The space of Borel probability measures on $\mathcal{N}[\mathbb{K}]$.
$W(a)$	(if $W \subseteq \mathbb{K}$ bounded measurable) the W-neighbourhood of a: the set of points $a + b$, where $b \in W$.
$W^{-1}(a)$	(if $W \subseteq \mathbb{K}$ bounded measurable) the set of points b where $a \in W(b)$.
$W(A)$	(if $W \subseteq \mathbb{K}$ bounded measurable and $A \subseteq \mathbb{K}$) the W-neighbourhood of A: the set of points $a + b$, where $a \in A$ and $b \in W$.
$C(\mathcal{X})$	The set of continuous functions $f : \mathbb{K} \to \mathbb{R}$.
$C_{\mathrm{c}}(\mathcal{X})$	The set of compactly supported continuous functions $f : \mathbb{K} \to \mathbb{R}$.
$BC(\mathcal{X})$	The set of bounded continuous functions $f : \mathbb{K} \to \mathbb{R}$.
$N_\varepsilon(a)$	The ε-ball around a.
λ	The intensity measure of the Poisson process or the fugacity measure of gas. A Radon measure on $\mathbb{K}$ (e.g., a multiple of the Lebesgue measure if $\mathbb{K} = \mathbb{R}^d$).
π^λ	The Poisson measure with intensity λ on $\mathcal{N}[\mathbb{K}]$.
P^λ	The Poisson specification with intensity λ.
δ_ω	Dirac measure concentrated at ω.
S	The set of particle species (e.g., the thin rod in d different directions).