Ergodicity of cellular automata subject to noise

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Computing in presence of noise

Challenge in building computers

When implementing computation using physical components,

- Transient errors due to thermal noise are inevitable.
- The smaller the scale, the more important the effect of noise.
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A simple model of (parallel) computation [cf. Turing machine]

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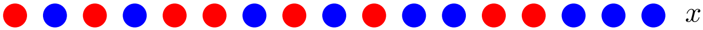

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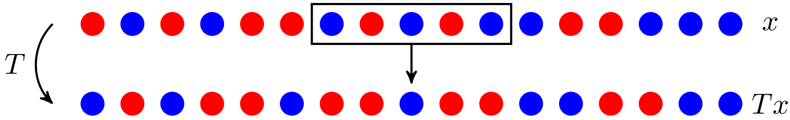

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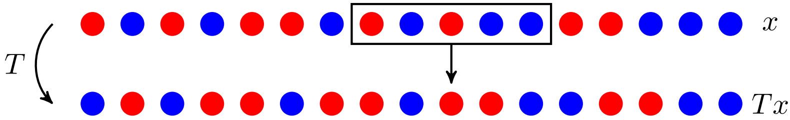

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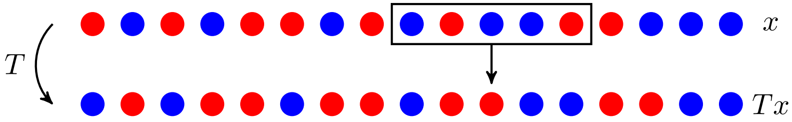

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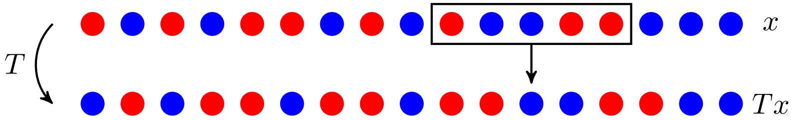

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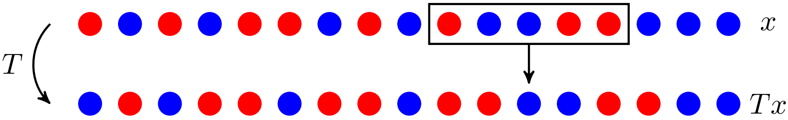

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- Iterate! [Same local rule at every time step!]

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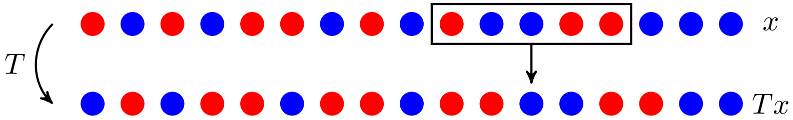

CA are discrete-time dynamical systems

- The set of all configurations x is a compact metric space!
- The global transformation $x \mapsto Tx$ is continuous!

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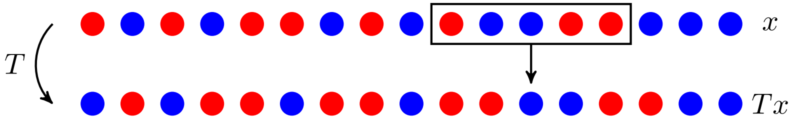

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- The set of all configurations x is a compact metric space!
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$\Longrightarrow$ We can exploit the machinery of dynamical systems and ergodic theory!

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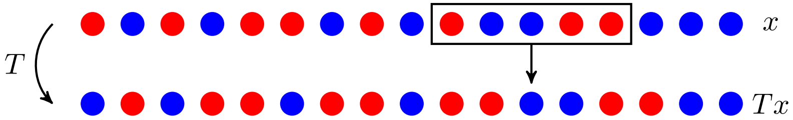

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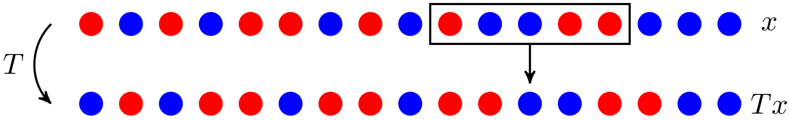

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- Finite number of possible states at each site
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- Reversibility and conservation laws can be easily implemented.

$\Longrightarrow$ Convenient for mathematical reasoning about physical implementations of computation.

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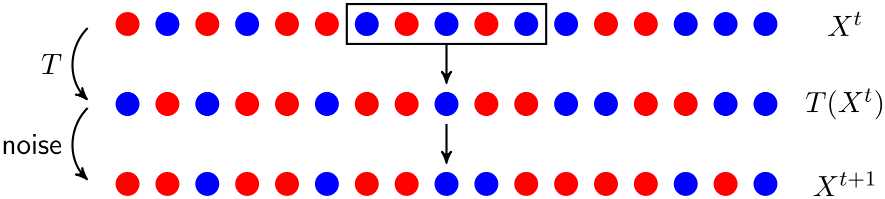

Cellular automata subject to noise

At each step,

a) first, apply the deterministic CA,

b) then, add noise independently at each site.

[Various models of noise possible!]

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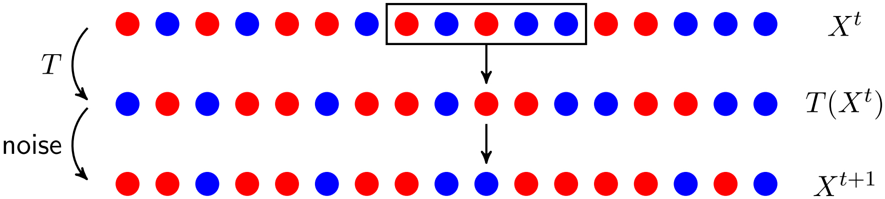

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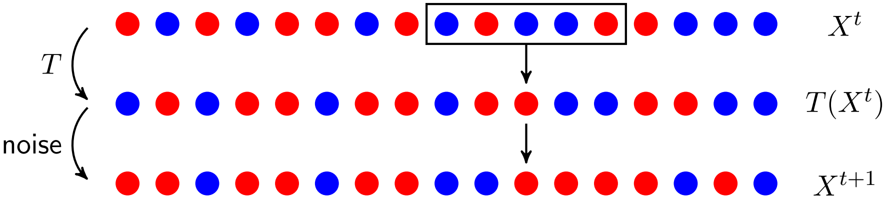

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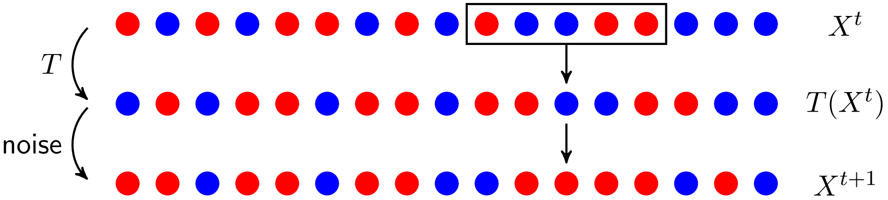

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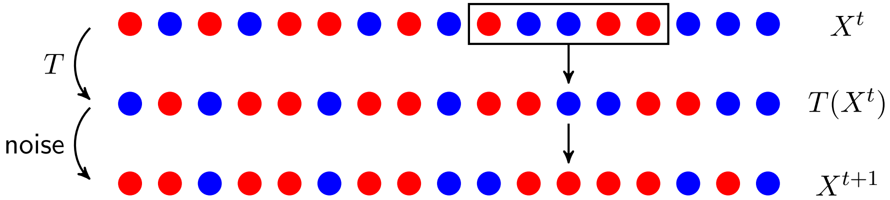

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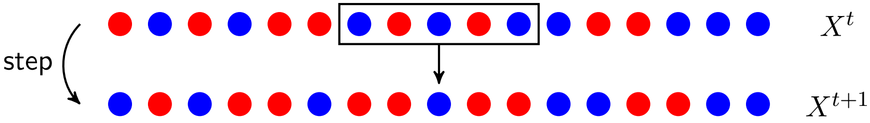
$\rightsquigarrow$ A special type of probabilistic cellular automaton (PCA).

Probabilistic cellular automata (PCA)



PCA are similar to CA, except that

- The local rule is probabilistic! [Described by a stochastic matrix]
- Symbols at different sites are updated independently.

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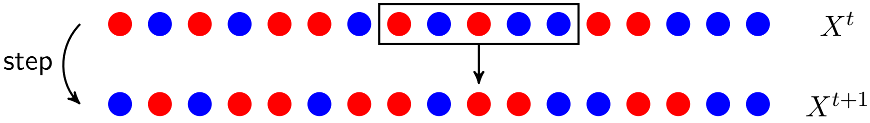

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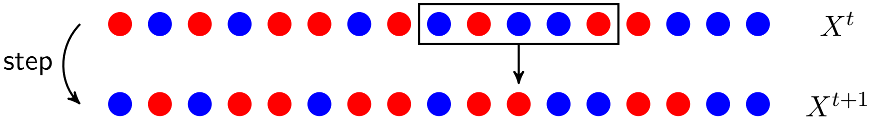

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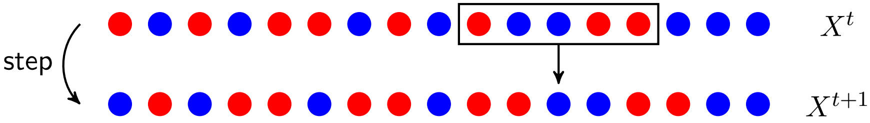

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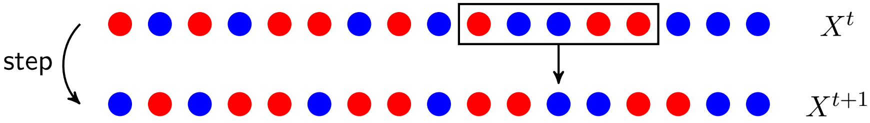

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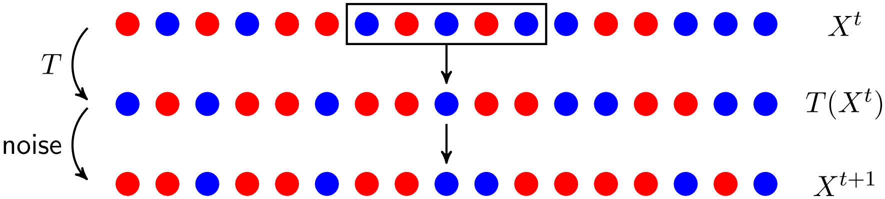
PCA are discrete-time Markov processes

- The state at time t is a random configuration X^t.
- The transition kernel has the Feller property.

[Discrete-time variants of interacting particle systems]

Computing with noisy CA



Problem (Reliable simulation)

Can we "simulate" a CA T with another CA S that is "reliable against sufficiently weak noise"?

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\text{encode}"'] & Tx \\ y \arrow[r] & Sy \arrow[u, "\text{decode}"'] \end{tikzcd}$$

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A simpler prerequisite:

Problem (Remembering a bit)

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Precise formulation in the language of Markov processes:

Problem (Ergodicity of noisy CA)

Find a CA that, in presence of sufficiently weak noise remains non-ergodic!

[Ergodicity: having a unique stationary measure that attracts every trajectory]

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  [Very sophisticated construction with astronomical number of symbols!]

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Problem (Sufficient conditions for ergodicity)

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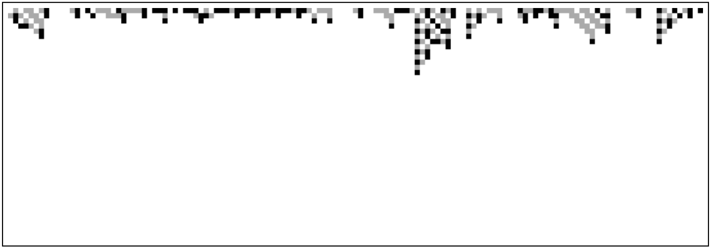
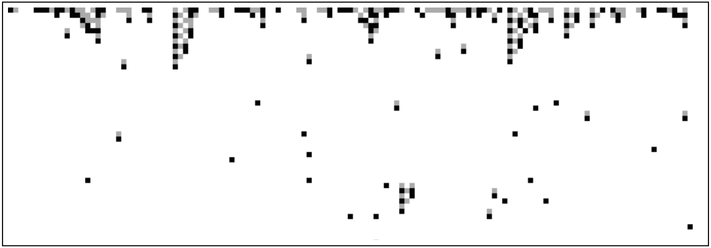
[A reliable CA should avoid those!]

Ergodicity of noisy CA

Example 1 (A nilpotent CA)





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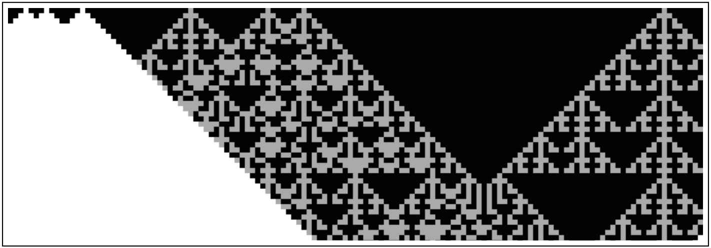
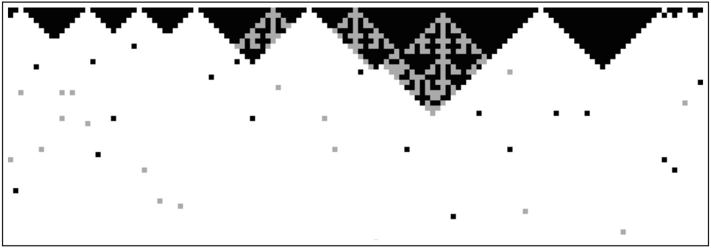
Example 2 (A CA with spreading symbol)





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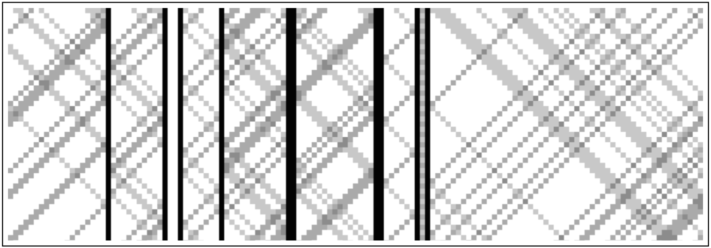
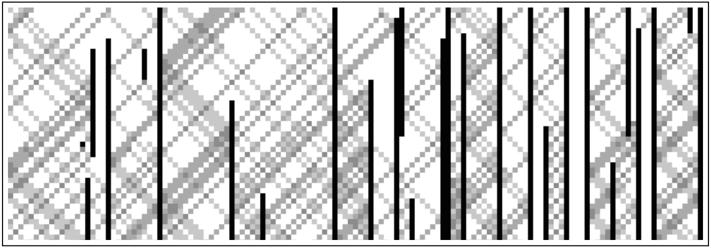
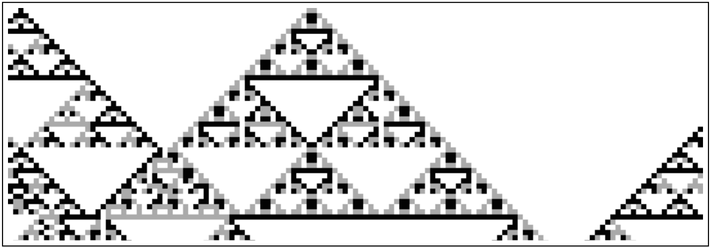
Example 3 (An almost equicontinuous CA)

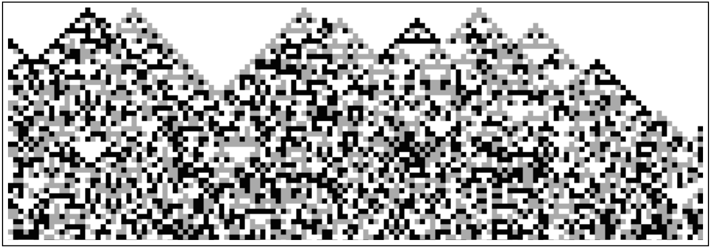




Ergodicity of noisy CA

Example 4 (A surjective CA)





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  [algorithmically undecidable?]
- Different mechanisms for ergodicity.

Ergodicity of noisy CA

Summary of results [Marcovici, Sablik, T. (2017)]

	Type of CA	**Type of noise**
I	Any CA	High noise
II	Nilpotent	Small perturbation
III	CA with spreading symbol	Memoryless noise
IV	———— " ———— (1d with $\mathcal{N} = \{0, 1\}$)	Small positive perturbation
V	Gliders with annihilation	Birth-death noise
VI	Simple gliders with reflecting walls	——— " ———
VII	Permutive	Permutation noise
VIII	Surjective	Additive noise
IX	XOR	Zero-range
X	Binary CA with spreading symbol	Zero-range (75% of parameter range)

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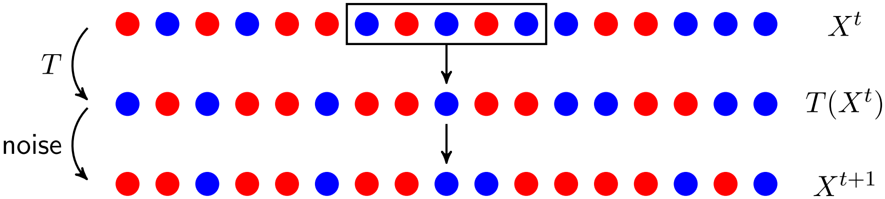

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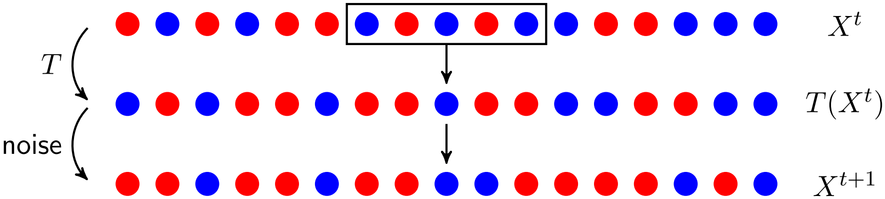

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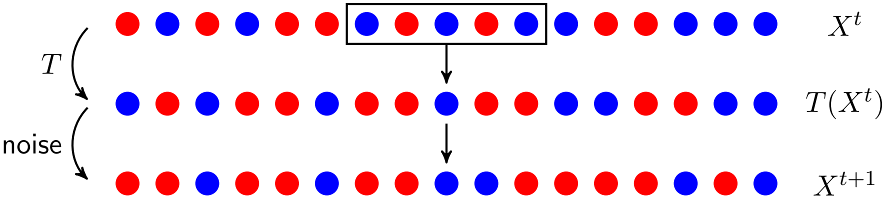
- Both a surjective CA and an additive noise preserve the uniform Bernoulli measure.

Surjective CA + additive noise



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A reversible CA-like computer subject to noise forgets all the information in its input/software exponentially fast!

Reversible computing with noisy components

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[see Aharonov, Ben-Or, Impagliazzo, Nisan (1996)]

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Practical implication

In order to implement noise-resilient (CA-like) computers, some degree of irreversibility is necessary.

[see Bennett (1982) and Bennett and Grinstein (1985)]

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Use entropy to measure the amount of information. □

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Proof ingredients.

a) A surjective CA does not "erase" entropy, only "diffuses" it.

b) Additive noise increases entropy. [Sharp estimate needed!]

For each finite set of sites J and each time step $t \geq 0$, we find

$$H(X_J^t) \geq \left[1 - (1-\kappa)^t\right] |J| \, \hbar - O(|\partial J|)$$

where $\hbar := \log |\Sigma|$ is the maximum capacity of a single site.

c) A bootstrap lemma

□

Surjective CA + zero-range noise

Theorem [Marcovici, T. (2018)]

A perturbation of a surjective CA with a positive zero-range noise is ergodic provided that both the CA and the noise preserve the same Bernoulli measure.

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Use pressure instead of entropy.
Use a characterization of when a surjective CA preserves a Bernoulli measure [Kari, T. (2015)]. □

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The pressure of a discrete random variable A w.r.t. an energy functional f is

$$\Psi_f(A) := H(A) - \mathbb{E}[f(A)] \, .$$

It can be thought of as a contorted version of entropy.

PCA with Bernoulli invariant measure

Theorem [Marcovici, T. (2018)]

Every positive-rate PCA that has a Bernoulli invariant measure is ergodic.

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  ⟶ Holley (1971), Holley and Stroock (1976) for IPS
  ⟶ Kozlov and Vasilyev (1980) for PCA
- With the exception of Holley and Stroock (1976), the entropy method has been limited to shift-invariant starting measures.

[Our result doesn't have this limitation.]

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Conjecture 1

Every positive-rate PCA that has a Gibbs invariant measure converges to the set of Gibbs measures with the same specification.

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Conjecture 2

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Entropy method for Markov processes

As a warm-up, consider the . . .

Convergence theorem of Markov chains

A finite-state Markov chain is ergodic
provided that it is irreducible and aperiodic.

[Convergence is exponentially fast!]

Different proofs

- Using Perron–Frobenius theory
- Using a coupling argument
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  [. . . for a suitable definition of conditional entropy $H(B \,|\, A)$]
- (continuity) $H(A)$ is continuous.

  [. . . as a function of the distribution of A]

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Let $X^0, X^1, \ldots$ be a Markov chain with finite state space Σ and transition matrix $\theta : \Sigma \times \Sigma \to [0, 1]$.

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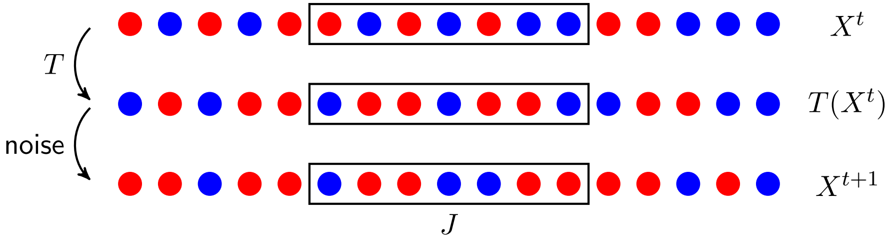
Proof of exponential convergence.

It follows from Fact II′ that

$$H(X^t) \geq \log|\Sigma| - \underbrace{(1-\kappa)^t \big[\log|\Sigma| - H(X^0)\big]}_{\to 0} \, . \qquad \square$$

Entropy method for surjective CA + additive noise



Note

- The uniform Bernoulli measure is stationary.
- In order to prove ergodicity, it is enough to show that for every finite set of sites J,

$$H(X_J^t) \to |J|\,\hbar \qquad \text{as } t \to \infty$$

where $\hbar := \log|\Sigma|$ is the maximum capacity of each site.

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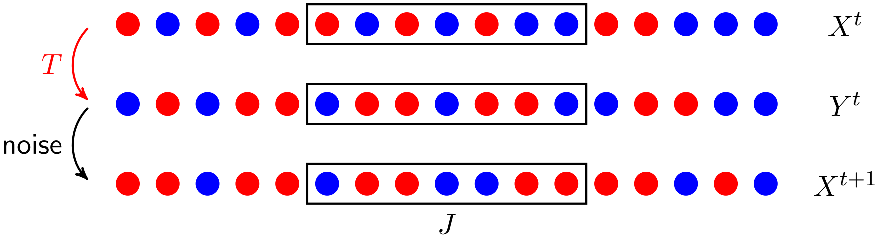

Effect of a surjective CA

A surjective CA does not "erase" entropy, only "diffuses" it:

$$H(Y^t_J) \geq H(X^t_J) - O(|\partial J|)$$

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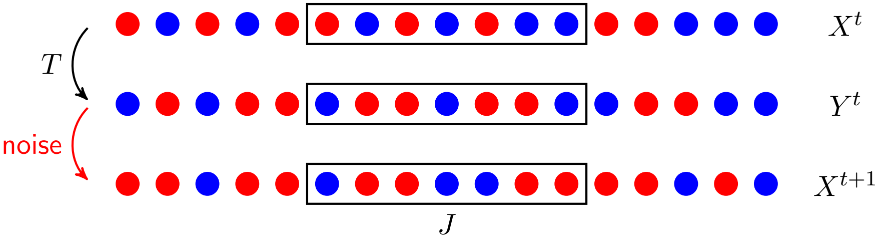

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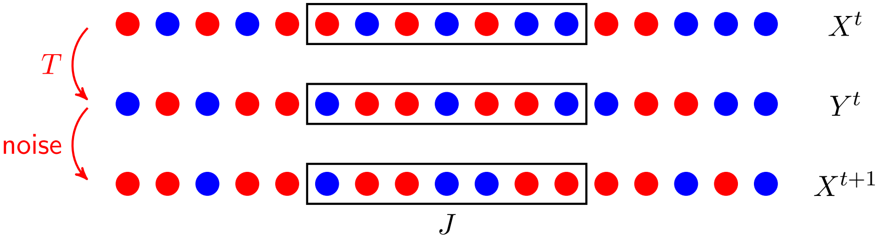

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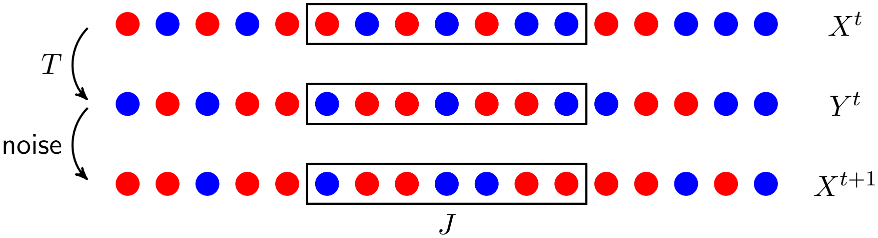

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for each $t \geq 0$.

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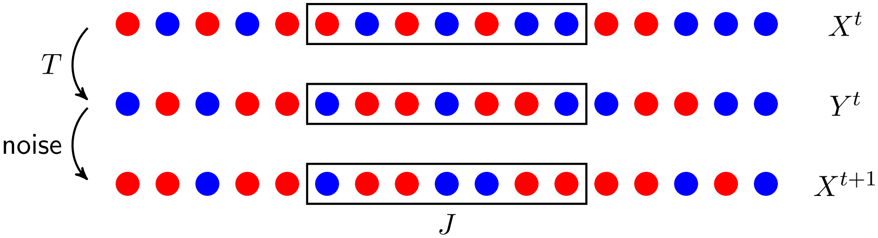

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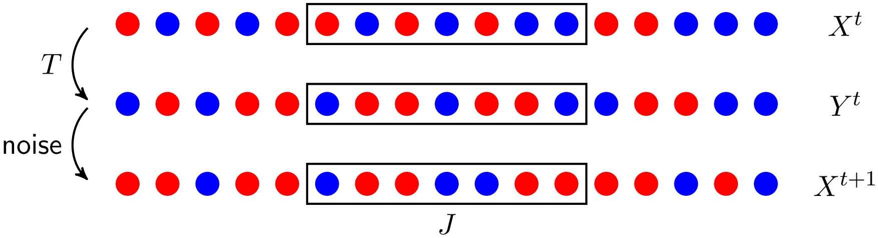

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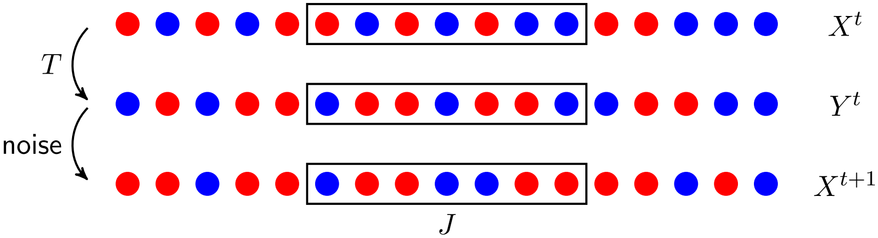

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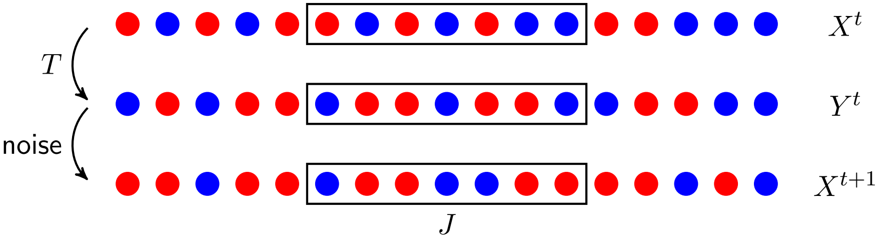

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A bootstrap lemma

The above implies ergodicity!

Intuitively:
Addition of entropy is much faster than its diffusion.

$\Longrightarrow$ entropy accumulates!

Summary

Key points

- Can a CA perform reliable computation in presence of noise?
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Thank you for your attention!