American University of Beirut

Introduction to Ergodic Theory

MATH 307K (2026–2027 Fall)

Assignment 3

Problem 1 (Proof of Birkhoff's ergodic theorem)**.** Complete the proof of Birkhoff's ergodic theory (i.e., remove the boundedness assumption) by following the approach sketched in the posted notes (or any other approach).

Problem 2 (Property of conditional expectation)**.** Let $(\mathcal{X}, \mathscr{F}_\mathcal{X}, \mu)$ and $(\mathcal{Y}, \mathscr{F}_\mathcal{Y}, \nu)$ be probability spaces and $\Phi\colon \mathcal{X} \to \mathcal{Y}$ a measurable map with $\Phi\mu = \nu$. Prove that

$$\nu(g \mid \mathscr{B}) \circ \Phi = \mu(g \circ \Phi \mid \Phi^{-1}\mathscr{B}) \qquad \mu\text{-a.e.}$$

for every function $g \in L^1_\nu(\mathcal{Y})$ and every sub-σ-algebra $\mathscr{B} \subseteq \mathscr{F}_\mathcal{Y}$.

Problem 3 (Limit in Birkhoff's theorem)**.** In class, it was claimed that the original statement of Birkhoff's ergodic theorem is equivalent to the following:

> **Theorem**. Let $(\mathcal{X}, \mathscr{F}, \mu)$ be a probability space and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. For every $f \in L^1_\mu(\mathcal{X})$, we have
>
> $$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \mu(f \mid \mathscr{I}_\mu)$$
>
> μ-almost everywhere as $n \to \infty$, where $\mathscr{I}_\mu$ is the sub-σ-algebra of invariant sets modulo μ.

Prove this equivalence.

Hint: For the forward direction, take an arbitrary $E \in \mathscr{I}_\mu$ and apply the original form of Birkhoff's theorem to $f \cdot \mathbb{1}_E$.

Problem 4 (Non-trivial isometries are not mixing)**.**

(a) Let $\mathcal{X} \coloneqq \mathbb{R}/\mathbb{Z}$. Let α be an irrational, and let $R_\alpha \coloneqq x + \alpha \pmod 1$ be the *rotation-by-α* map on $\mathcal{X}$. Prove that R_α is not weakly mixing.

(b) Let $T\colon \mathcal{X} \to \mathcal{X}$ be an isometry of a metric space $\mathcal{X}$, and let μ be a T-invariant probability measure that is not just a point mass at a fixed point of T. Prove that T is not weakly mixing.

Problem 5 (Weak mixing and Cartesian product)**.** Prove that the product of an ergodic system with any weakly mixing system is ergodic.

Problem 6 (Cesàro convergence vs. convergence in density)**.** In class, we stated the following proposition as a corollary of the "full-density diagonal sequence" lemma:

> **Proposition**. Let $a_0, a_1, \ldots$ be a sequence of real numbers from a bounded interval $I = [C, D]$. The following conditions are equivalent:
>
> (i) (Cesàro convergence to an extremal value) $\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} a_k \to C$ as $n \to \infty$.
>
> (ii) (convergence in density) There exists a full-density set $J \subseteq \mathbb{N}$ such that $a_n \to C$ as $J \ni n \to \infty$.

(a) Prove this proposition.

(b) Provide a counter-example showing that a similar equivalence does not hold if $a_0, a_1, \ldots$ are from an unbounded interval $I = [C, \infty)$.

(c) Provide a counter-example showing that a similar equivalence does not hold if the limit is an interior point $x \in (C, D)$ rather than C.

Problem 7 (Covergence in density vs. statistical convergence)**.** Let $(\mathcal{X}, \rho)$ be a metric space. Prove that for every $a_0, a_1, a_2, \ldots \in \mathcal{X}$ and $a \in \mathcal{X}$, the following conditions are equivalent:

(i) (statistical convergence) For every $\varepsilon > 0$, we have $\displaystyle\lim_{n\to\infty} \frac{|\{k < n : \rho(a_k, a) \geq \varepsilon\}|}{n} = 0$.

(ii) (convergence in density) There exists a set $J \subseteq \mathbb{N}$ of full density such that $\displaystyle\lim_{J \ni n \to \infty} a_n = a$.