Minimality and almost periodicity

Assumption. We restrict ourselves to topological systems where the underlying system is compact and metrizable.

Minimality is another analogue of ergodicity in topological systems. Recall that a subsystem of a topological system $(\mathcal{X}, T)$ is a system $(\mathcal{Y}, S)$ where $\mathcal{Y} \subseteq \mathcal{X}$ and $S = T|_{\mathcal{Y}}$.

Definition 1 (Transitive system)**.** A topological system $(\mathcal{X}, T)$ is said to be minimal if it does not have any proper closed subsystem.

In other words, $(\mathcal{X}, T)$ is minimal if every topologically closed, forward invariant subset of $\mathcal{X}$ is either empty or the entire $\mathcal{X}$.

A general way to make subsystems. Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metric space. Let $z \in \mathcal{X}$ and consider

$$E \coloneqq \overline{\mathcal{O}^+(z)}\,.$$

Then, E is closed and $TE \subseteq E$, hence defines a subsystem of $(\mathcal{X}, T)$. This is called the *orbit closure* of z.

Note that $z \in \mathcal{X}$ is a transitive point for T if and only if its orbit closure is $\mathcal{X}$.

Theorem 2 (Characterizations of minimality)**.** *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$. The following conditions are equivalent:*

(a) T is minimal.

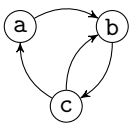
(b) Every point $x \in \mathcal{X}$ is a transitive point for T.

(c) For every non-empty open set $A \subseteq \mathcal{X}$, we have $\bigcup_{n\geq 1} T^{-n}A = \mathcal{X}$.

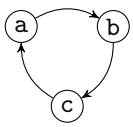
Proof. Exercise. □

Example 3 (Irrational rotations)**.** Every irrational rotation $R_\alpha\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is minimal. Indeed, by Kronecker's theorem, every point $x \in \mathbb{R}/\mathbb{Z}$ is transitive for R_α. ○

Example 4 (Vertex shifts)**.** A vertex shift is minimal if and only its underlying graph consists of a single cycle.



$x = \texttt{bcbcbc}\cdots$ is not transitive.



minimal

○

Example 5 (Minimal subshifts)**.** Let Σ be a finite alphabet. The *language* of a subshift $\mathcal{X} \subseteq \Sigma^{\mathbb{N}}$ is the set

$$L(\mathcal{X}) \coloneqq \{w \in \Sigma^* : x \in [w] \text{ for some } x \in X\}$$

of all finite words that appear in some configuration in $\mathcal{X}$. A subshift $\mathcal{X}$ is minimal if and only if every word $w \in L(\mathcal{X})$ appears in every configuration $x \in \mathcal{X}$ infinitely many times. ○

Notes for Introduction to Ergodic Theory (MATH 307K), Siamak Taati
Last update: October 28, 2026

Example 6 (XOR cellular automaton)**.** The XOR cellular automaton is not minimal. For instance, the configuration

$$x = \cdots \mathtt{000} \cdots$$

is a fixed point, hence not transitive. ○

Example 7 (Dyadic adding machine)**.** The dyadic adding machine $T\colon \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}} \to \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ is minimal.

		1	2	3	4	$\cdots$
x	$=$	0	0	0	0	$\cdots$
$T(x)$	$=$	1	0	0	0	$\cdots$
$T^2(x)$	$=$	0	1	0	0	$\cdots$
$T^3(x)$	$=$	1	1	0	0	$\cdots$
$T^4(x)$	$=$	0	0	1	0	$\cdots$
$T^5(x)$	$=$	1	0	1	0	$\cdots$
$T^6(x)$	$=$	0	1	1	0	$\cdots$
$T^7(x)$	$=$	1	1	1	0	$\cdots$
$T^8(x)$	$=$	0	0	0	1	$\cdots$

Indeed, let $x \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ and $k \in \mathbb{N}$. Observe that the sequence

$$x_k, T(x)_k, T^2(x)_k, \ldots$$

is periodic with period 2^{k+1}. More generally, for every $\ell > 0$, the sequence

$$x_{[0,\ell)}, T(x)_{[0,\ell)}, T^2(x)_{[0,\ell)}, \ldots$$

is periodic with period 2^ℓ, and cycles through all words of length ℓ. In particular, for every word $u = u_0u_1\cdots u_{\ell-1}$, exactly one of the configurations $T(x), T^2(x), T^3(x), \ldots, T^{2^\ell}(x)$ belongs to the cylinder $[u]$. Therefore, x is transitive. ○

In the latter example, every point behaves almost like a periodic point x in the sense that, for every open ball $B \ni x$, the orbit of x visits B periodically. This is a strong form of recurrence.

Definition 8. Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a topological system. We say that

- A point $x \in \mathcal{X}$ is a *recurrent point* if, for every open $B \ni x$, there exists an $n \geq 1$ such that $T^n(x) \in B$.

  Exercise. Show that in this case, there are infinitely many such n's.

- A point $x \in \mathcal{X}$ is *almost periodic* if, for every open $B \ni x$, there exists a $p > 0$ such that for every $m \geq 0$, $T^n(x) \in B$ for some n satisfying $m \leq n < m + p$.

  (In other words, this means that $\{n \in \mathbb{N} : T^n(x) \in B\}$ has bounded gaps.)

- A point $x \in \mathcal{X}$ is *quasiperiodic* if, for every open $B \ni x$, there exists a $p > 0$ such that $T^{ip}(x) \in B$ for every integer $i \geq 0$.

Thus, every point in the dyadic adding machine is quasiperiodic. Clearly,

$$x \text{ is periodic} \implies x \text{ is quasiperiodic} \implies x \text{ is almost periodic} \implies x \text{ is recurrent.}$$

Example 9 (Irrational rotations)**.** Under an irrational rotation, every point is almost periodic but not quasiperiodic.

Indeed, let us show that every $x \in \mathbb{R}/\mathbb{Z}$ is almost periodic under every rotation $R_\alpha\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with irrational α. Let (a,b) be an open interval containing x. Let m be such that $\ell := d\big(0, R_\alpha^m(0)\big) < b - a$. (Here, d denotes the standard metric on $\mathbb{R}/\mathbb{Z}$.) Then, the orbit of each point $y \in (a,b)$ under R_α^m revisits (a,b) within $\lceil 1/\ell \rceil$ steps. Thus, the orbit of x under R_α revisits (a,b) at least once every $p := m\lceil 1/\ell \rceil$ steps. Therefore, x is almost periodic.

Showing that every point is not quasiperiodic is left as an exercise. ○

Theorem 10 (Birkhoff)**.** *In a minimal system, every point is almost periodic.*

Proof. Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$, and suppose there is a point $z \in \mathcal{X}$ that is not almost periodic under T. Then, there exists an open neighbourhood $B \ni z$ for which the return times are not almost periodic, that is,

- For every $p > 0$, there exists an $m_p \in \mathbb{N}$ such that $T^n(x) \notin B$ for all n with $m_p \le n < m_p + p$.

Now, consider

$$\begin{array}{lllllll} T^{m_1}(x) & & & & & & \\ T^{m_2}(x) & \longrightarrow & T^{m_2+1}(x) & & & & \\ T^{m_3}(x) & \longrightarrow & T^{m_3+1}(x) & \longrightarrow & T^{m_3+2}(x) & & \\ T^{m_4}(x) & \longrightarrow & T^{m_4+1}(x) & \longrightarrow & T^{m_4+2}(x) & \longrightarrow & T^{m_4+3}(x) \\ \vdots & & \vdots & & \vdots & & \vdots \qquad \ddots \end{array}$$

All these points belong to the closed (hence, compact) set $\mathcal{X} \setminus B$. By compactness, there exist $p_1 < p_2 < p_3 < \cdots$ and $x \in \mathcal{X} \setminus B$ such that $T^{m_{p_i}}(z) \to x$ as $i \to \infty$.

Now, consider $k \ge 0$. Since T^k is continuous, $T^{m_{p_i}+k}(z) \to T^k(x)$ as $i \to \infty$. Since $T^{m_{p_i}+k}(z) \in \mathcal{X} \setminus B$ for all large i and $\mathcal{X} \setminus B$ is closed, we find that $T^k(x) \in \mathcal{X} \setminus B$.

The latter is true for every $k \ge 0$. In other words, the entire orbit of x is in the closed subset $\mathcal{X} \setminus B$. It follows that $\overline{\mathcal{O}^+(x)} \subseteq \mathcal{X} \subseteq B \subsetneq \mathcal{X}$. We have found a proper subsystem of $(\mathcal{X}, T)$. Therefore, $(\mathcal{X}, T)$ is not minimal. □

Exercise 11. Let $(\mathcal{X}, T)$ be a topological system and $z \in \mathcal{X}$. The above theorem implies that that:

- If $\overline{\mathcal{O}^+(z)}$ is minimal, then z is almost periodic.

Prove the converse:

- If z is almost periodic, then $\overline{\mathcal{O}^+(z)}$ is minimal.

Proposition 12. *Every topological system has a minimal subsystem.*

Proof. The proof is via Zorn's lemma.

Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$. Consider the family $\mathscr{C}$ of all non-empty, closed, forward invariant sets $E \subseteq \mathcal{X}$. This family is partially ordered with inclusion.

Claim. Every linearly ordered subfamily $\mathscr{A} \subseteq \mathscr{C}$ has a lower bound in $\mathscr{C}$.

⌜ *Argument.* Let $W \coloneqq \bigcap_{E \in \mathscr{A}} E$. Note that

- $W \subseteq E$ for every $E \in \mathscr{A}$.
- W is a closed subset of $\mathcal{X}$.
- $T(W) \subseteq W$.
  Indeed, if $x \in W$, then $x \in E$ for every $E \in \mathscr{A}$, hence $T(x) \in E$ because E is forward invariant. It follows that $T(x) \in W$.
- W is non-empty, by compactness.

⌞ Therefore, W is an element of $\mathscr{C}$ that is a lower bound for $\mathscr{A}$.

Now, by Zorn's lemma, $\mathscr{C}$ has a minimal element E_0. This minimal element defines a minimal subsystem of $(\mathcal{X}, T)$. □

Corollary 13 (Birkhoff's recurrence theorem)**.** *Every topological system, has at least one almost periodic (in particular, recurrent) point.*

Let us restate the result of Exercise 11 as a theorem.

Theorem 14. *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space. A point $z \in \mathcal{X}$ is almost periodic under T if and only if its orbit closure $\overline{\mathcal{O}^+(z)}$ forms a minimal subsystem.*

Example 15 (Toeplitz subshifts)**.** Let Σ be a finite alphabet. A non-periodic quasiperiodic configuration $z \in \Sigma^{\mathbb{N}}$ is called a *Toeplitz sequence*. The orbit closure of such a configuration in the full shift $\Sigma^{\mathbb{N}}$ is called a *Toeplitz subshift*.

Here is an example of a Toeplitz sequence with alphabet $\Sigma \coloneqq \{\texttt{[}, \texttt{]}\}$. To construct z, start with placing pairs of brackets at every other position. We are left with holes at positions $2i+1$ with $i \in \mathbb{N}$. Next, fill up every other hole with pairs of brackets. We are left with holes at positions $4i+3$ with $i \in \mathbb{N}$. Again, fill up every other remaining hole with pairs of brackets. Repeat this process indefinitely to fill up every hole.



The resulting configuration is a Toeplitz sequence, known as the *paper-folding* sequence. ○

Exercise 16. Prove that a non-periodic configuration $z \in \Sigma^{\mathbb{N}}$ is Toeplitz if and only if $\mathbb{N}$ can be partitioned into infinite arithmetic progressions $P_1, P_2, \ldots$ such that z is constant over P_i for each i.