More on topological recurrence

Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$, and let μ be a Borel probability measure on $\mathcal{X}$ that is T-invariant. Recall the following two theorems:

Theorem (Birkhoff's recurrence theorem)**.** *There exists a point $z \in \mathcal{X}$ that is almost periodic (in particular, recurrent) under T.*

Theorem (Poincaré's recurrence theorem)**.** *If $A \subseteq \mathcal{X}$ is measurable with $\mu(A) > 0$, then the orbit of μ-almost every $x \in A$ returns to A infinitely many times.*

We will now prove a topological version of Poincaré's theorem:

Theorem (Poincaré's topological recurrence theorem; version 1)**.** *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$ and μ a T-invariant Borel probability measure on $\mathcal{X}$. Then, μ-almost every point in $\mathcal{X}$ is (topologically) recurrent.*

As we shall see later in the course, every topological system admits at least one invariant probability measure. Thus, Poincaré's topological recurrence theorem implies a weak form of Birkhoff's theorem: *Every topological system has at least one recurrent point.*

Proof of Poincaré topological recurrence theorem; version 1. Recall that every compact metrizable space has a countable topological base. Let $A_1, A_2, \dots$ be a sequence of non-empty open sets that form a base. Thus, every open subset of $\mathcal{X}$ is a union of some of these sets.

For each k, let

$$E_k \coloneqq \{x \in A_k : T^n(x) \notin A_k \text{ for every } n \geq 1\} = A_k \setminus \bigcup_{n\geq 1} T^{-n} A_k$$

be the set of those points in A_k whose trajectories do not return to A_k. by Poincaré's first theorem, $\mu(E_k) = 0$. Note that this is true irrespective of whether $\mu(A_k) = 0$ or $\mu(A_k) > 0$. Let $\mathcal{R} \coloneqq \mathcal{X} \setminus \bigcup_{k=1}^{\infty} E_k$. Observe that

$$\mu(\mathcal{R}) = 1 - \mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \geq 1 - \sum_{k=1}^{\infty} \mu(E_k) = 1\,,$$

hence $\mu(\mathcal{R}) = 1$. We claim that every point $x \in \mathcal{R}$ is topologically recurrent.

Indeed, let $B \ni x$ be an open set. Since A_k's form a base, there exists a k such that $x \in A_k \subseteq B$. Since $x \notin E_k$, there exists an $n \geq 1$ such that $T^n(x) \in A_k \subseteq E_k$. We conclude that x is recurrent. □

Definition (Topological support)**.** Let μ a positive Borel measure on a topological space $\mathcal{X}$. The (topological) *support* of μ is the set

$$\operatorname{supp}(\mu) \coloneqq \{x \in \mathcal{X} : \mu(B) > 0 \text{ for every open } B \ni x\}\,.$$

We say that μ has *full support* if $\operatorname{supp}(\mu) = \mathcal{X}$.

Observe that

$$\operatorname{supp}(\mu) = \mathcal{X} \setminus \bigcup_{\substack{B \text{ open}\\ \mu(B)=0}} B = \bigcap_{\substack{B \text{ open}\\ \mu(B)=0}} (\mathcal{X} \setminus B) = \bigcap_{\substack{C \text{ closed}\\ \mu(\mathcal{X}\setminus C)=0}} C \overset{\text{if } \mu \text{ is a probability measure}}{=} \bigcap_{\substack{C \text{ closed}\\ \mu(C)=1}} C\,.$$

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Exercise. Let μ be a probability measure on a compact metrizable space. Show that $\mu\big(\operatorname{supp}(\mu)\big) = 1$.

Exercise. Let μ be a full-support probability measure on a compact metrizable space $\mathcal{X}$. Show that every measurable set $E \subseteq \mathcal{X}$ with $\mu(E) = 1$ is dense.

When a system admits a full-support invariant measure, Poincaré's theorem gives a stronger conclusion:

Theorem (Poincaré's topological recurrence theorem; version 2)**.** *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$. Suppose that T admits a full-support invariant measure. Then, the set of (topologically) recurrent points is residual.*

Proof. Following the proof of the first version, it is enough to verify that, for each k, the set E_k is nowhere dense.

Since $T^{-n}A_k$ is open for every $n \geq 1$, we have

$$\overline{E_k} \subseteq \partial A \cup E_k \ .$$

To show that E_k is nowhere dense, it is enough to show that $\partial A \cup E_k$ has no interior. Note that

- ∂A is the boundary of an open set, hence has no interior.
- $\mu(E_k) = 0$ for a full-support μ, hence E_k has no interior.

Now, let $x \in \operatorname{int}(\partial A \cup E_k)$. Since E_k has no interior point, every open neighbourhood $B \ni x$ must intersect ∂A. Since ∂A is closed, it follows that $x \in \partial A$. We conclude that $\operatorname{int}(\partial A \cup E_k) \subseteq \partial A$. But we know that ∂A has no interior. Therefore, $\operatorname{int}(\partial A \cup E_k) = \varnothing$. □