Topological transitivity

Recall that a measure-preserving transformation $T\colon \mathcal{X} \to \mathcal{X}$ of a probability space $(\mathcal{X}, \mu)$ is ergodic if and only if for every two measurable sets $A, B \subseteq \mathcal{X}$ with $\mu(A), \mu(B) > 0$, there exists an $n \geq 0$ such that $\mu(A \cap T^{-n}B) > 0$.

Definition (Transitive system)**.** A continuous transformation $T\colon \mathcal{X} \to \mathcal{X}$ of a topological space $\mathcal{X}$ is said to be (topologically) (forward) *transitive* if for every two non-empty open sets $A, B \subseteq \mathcal{X}$, there exists an $n \geq 0$ such that $A \cap T^{-n}B \neq \varnothing$.

Both properties indicate that the system is "indecomposable", one in measurable sense, the other in topological sense.

Example.

- Irrational rotations of $\mathbb{R}/\mathbb{Z}$ are transitive; rational rotations are not.
- Every full shift is transitive.
- The golden mean shift is transitive. ○

The latter two examples have a stronger property:

Definition (Topological mixing)**.** A continuous transformation $T\colon \mathcal{X} \to \mathcal{X}$ of a topological space $\mathcal{X}$ is said to be *topologically mixing* if for every two non-empty open sets $A, B \subseteq \mathcal{X}$, there exists an $n_0 \geq 0$ such that $A \cap T^{-n}B \neq \varnothing$ for all $n \geq n_0$.

Example.

- Irrational rotations of $\mathbb{R}/\mathbb{Z}$ are not topologically mixing.
- Every full shift is topologically mixing.
- The golden mean shift is topologically mixing. ○

Proposition. *Let $G = (V, E)$ be a finite directed graph.*

(a) *The vertex/edge shift associated to G is transitive if and only if G is strongly connected.*

(b) *The vertex/edge shift associated to G is topologically mixing if and only if G is strongly connected and aperiodic.*

Exercise. Show that the XOR cellular automaton $T\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ defined by

$$(Tx)_i \coloneqq (x_i + x_{i+1}) \bmod 2$$

is topologically mixing.
Hint: T is additive with respect to pointwise addition: $T(x+y) = T(x) + T(y)$. ○

Example. The dyadic adding machine $T\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ defined by

$$(Tx)_i \coloneqq \begin{cases} 0 & \text{if } x_0 = x_1 = \cdots = x_i = 1, \\ 1 & \text{if } x_0 = x_1 = \cdots = x_{i-1} = 1 \text{ and } x_i = 0, \\ x_i & \text{otherwise,} \end{cases}$$

is transitive but not mixing. ○

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Definition (Transitive point)**.** Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a topological space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a (forward) *transitive point* if, for every non-empty open set $A \subseteq \mathcal{X}$, there exists an $n \geq 1$ such that $T^n(x) \in A$.

Using the notation

$$\mathcal{O}(x) \coloneqq \{T^n(x) : n \geq 0\} \qquad \text{(orbit of } x\text{)}$$
$$\mathcal{O}^+(x) \coloneqq \{T^n(x) : n \geq 1\}$$

we can say that a point $x \in \mathcal{X}$ is transitive if and only if $\mathcal{O}^+(x)$ is dense.

Example. Consider the full shift $(\Sigma^{\mathbb{N}}, \sigma)$. Let

$$u^0, u^1, u^2, \ldots$$

be an enumeration of all non-empty elements of Σ^*. The configuration

$$z = u^0 u^1 u^2 \cdots$$

obtained by concatenating these words is a transitive point for σ. For instance, if $\Sigma = \{0, 1\}$, then

$$z = 0\,1\,00\,01\,10\,11\,000\,\cdots$$

is a transitive point. ○

Exercise. Find a transitive point in the golden mean shift.

Example. In the definition of a transitive point, the requirement that n is not zero is important. For instance, consider the finite system $\mathcal{X} \coloneqq \{0, 1\}$ with $T(0) = T(1) = 1$. This system has no transitive point, but 0 satisfies the following weaker condition: For every non-empty open set $A \subseteq \mathcal{X}$, there exists an $n \geq 0$ such that $T^n(0) \in A$. ○

By definition, the orbit of a transitive point x visits every non-empty open set A. In fact, this happens infinitely many times.

Proposition. *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a topological space $\mathcal{X}$. Let x be a transitive point in a topological system $(\mathcal{X}, T)$ and $A \subseteq \mathcal{X}$ a non-empty open set. Then, for every $m \in \mathbb{N}$, there exists an $n \geq m$ such that $T^n(x) \in A$.*

An intuitive (but buggy) proof. Let $n_1 > 0$ be the first time such that $T^{n_1}(x) \in A$. The set $A' \coloneqq A \setminus \{T^{n_1}(x)\}$ is open, so there exists an $n_2 > 0$ such that $T^{n_2}(x) \in A' \subseteq A$. Clearly, $n_2 > n_1$. We can now repeat this process to generate an infinite sequence $n_1 < n_2 < n_3 < \cdots$ such that $T^{n_k}(x) \in A$ for $k = 1, 2, 3, \ldots$. □

Exercise. Where is the bug in the above proof?

Exercise. Show that any continuous transformation with a transitive point is surjective.

Proof of the proposition. Let $m \in \mathbb{N}$. Since T is continuous and surjective, $T^{-m}(A)$ is open and non-empty. Therefore, by the transitivity of x, there exists an $\ell \geq 1$ such that $T^\ell(x) \in T^{-m}(A)$. It follows that $T^{m+\ell}(x) \in A$, hence the claim holds for $n \coloneqq m + \ell$. □

Theorem (Characterizations of transitivity)**.** *Let $T\colon \mathcal{X} \to \mathcal{X}$ be a continuous transformation of a compact metrizable space $\mathcal{X}$. The following conditions are equivalent:*

(a) *T is topologically transitive.*

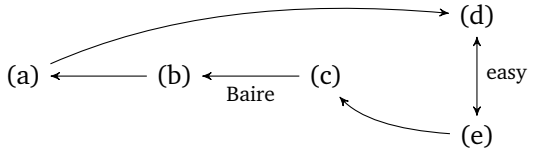
(b) *T has a transitive point.*

(c) *The set of transitive points is residual.*

(d) *If $E \subseteq \mathcal{X}$ is closed and forward invariant, then either $E = \mathcal{X}$ or E is nowhere dense.*

(e) *If $F \subseteq \mathcal{X}$ is open and backward invariant, then either $F = \varnothing$ or F is dense.*

Proof. The equivalence of (d) and (e) is clear. The implication (c)⇒(b) follows from Baire's category theorem (*Every residual subset of a compact metrizable space is non-empty*). We prove the remaining implications in the following diagram:



- (b)⇒(a):

  Suppose that T has a transitive point x. Let $A, B \subseteq \mathcal{X}$ be non-empty open sets. Since x is transitive, there exists $m \geq 1$ such that $T^m(x) \in A$. By the latter proposition, there also exists an $n \geq m$ such that $T^n(x) \in B$. Set $z \coloneqq T^m(x)$. Note that $z \in A$ and $T^{n-m}(z) \in B$, hence $A \cap T^{-(n-m)}B \neq \varnothing$. Therefore, T is transitive.

- (a)⇒(d):

  We prove the contrapositive. Let $E \subseteq \mathcal{X}$ be a closed set with $TE \subseteq E$ and suppose that $E \neq \mathcal{X}$ and $\mathring{E} \neq \varnothing$. Let $A \coloneqq \mathring{E}$ and $B \coloneqq \mathcal{X} \setminus E$. Observe that A and B are non-empty. On the other hand, for every $x \in A$ and $n \geq 0$, we have $T^n(x) \in T^n(E) \subseteq E$, hence $T^n(x) \neq B$. Therefore, T is not transitive.

- (e)⇒(c):

  Suppose that (e) holds.

  Let us first show that T is surjective. Indeed, suppose that T is not surjective. Let

  $$F \coloneqq \{x : x \text{ has no pre-image under } T\} = \mathcal{X} \setminus T\mathcal{X} \, .$$

  Since $\mathcal{X}$ is compact, F is open. Furthermore, $T^{-1}F = \varnothing \subseteq F$, which means F is backward invariant. Since F is non-empty, it follows that F is dense. Now, let us consider two cases:
  Case 1: F is singleton. But this means $\mathcal{X} = F$, and $T^{-1}\mathcal{X} = \varnothing$, which is a contradiction.
  Case 2: F has two distinct elements a, b. Let $A \subseteq \mathcal{X}$ be an open neighbourhood of a such $\overline{A} \not\ni b$. That such a set exists follows from the fact that $\mathcal{X}$ is compact and metrizable (or, more abstractly, from the regularity of $\mathcal{X}$). Set $F' \coloneqq F \setminus \overline{A}$. Then, F' is again open and backward invariant. But F' is neither empty, nor dense, contradicting (e).

  Let us now show that the set of transitive points for T is residual. Let $A_1, A_2, \ldots$ be a countable base for the topology of $\mathcal{X}$. (*Exercise:* every compact metrizable space has a countable base.) Without loss of generality, we assume that $A_k \neq \varnothing$ for each k. For each k, let

  $$B_k \coloneqq \bigcup_{n \geq 1} T^{-n} A_k = \{x \in \mathcal{X} : \mathcal{O}^+(x) \cap A_k \neq \varnothing\} \, ,$$

  and set $\mathcal{T} \coloneqq \bigcap_{k=1}^{\infty} B_k$. The latter is precisely the set of transitive points for T.

  Note that B_k is open and backward invariant. Since T is surjective, B_k is non-empty. Hence, it must be dense. Therefore, $\mathcal{T}$ is the intersection of countably many dense open sets, which means it is a residual set. □