Kac's recurrence theorem

Recall Poincaré's theorem for measure-preserving dynamical systems:

Theorem (Poincaré's Recurrence Theorem; 1890). *Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$. Then, for μ-a.e. $x \in A$, we have $T^n(x) \in A$ for some $n \geq 1$. In fact, the orbit of μ-a.e. $x \in A$ returns to A infinitely many times.*

Kac's recurrence theorem provides quantitative information on Poincaré's recurrence times. For a measurable set $A \subseteq \mathcal{X}$ and a point $x \in \mathcal{X}$, we define

$$\begin{aligned} \tau_A(x) &\coloneqq \inf\{n \geq 0 : T^n(x) \in A\} && \text{(the \textit{first hitting time} of } A) , \\ \tau_A^+(x) &\coloneqq \inf\{n > 0 : T^n(x) \in A\} && \text{(the \textit{first return time} to } A) . \end{aligned}$$

Observe that

- $\tau_A(x) = \tau_A^+(x)$ for $x \notin A$.
- $\tau_A(x) \in \mathbb{N} \cup \{\infty\}$ and $\tau_A^+(x) \in \mathbb{Z}^+ \cup \{\infty\}$.
- $\tau_A(x) < \infty$ means the orbit of x passes A.
- Assuming $\mu(A) > 0$, we have $\tau_A^+(x) < \infty$ for μ-a.e. $x \in A$.

Theorem (Kac's Recurrence Theorem; 1947). *Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$. Then,*

$$\int_A \tau_A^+ \,\mathrm{d}\mu = \mu(\widetilde{A}) \leq 1 ,$$

where

$$\widetilde{A} \coloneqq \{x \in \mathcal{X} : T^n(x) \in A \text{ for some } n \geq 0\} = \{x \in \mathcal{X} : \tau_A(x) < \infty\}$$

is the set of points whose orbits pass A.

Interpretation. The statement becomes clearer if we rewrite it as follows:

$$\frac{1}{\mu(A)} \int_A \tau_A^+ \,\mathrm{d}\mu = \frac{\mu(\widetilde{A})}{\mu(A)} \leq \frac{1}{\mu(A)} .$$

Note that the left-hand side is simply the "mean return time to A among all $x \in A$", where the mean is taken with respect to μ. Alternatively, in the probabilistic language, the left-hand side is the conditional expectation of the return time to A given that the initial state is in A.

Before we proceed with the proof, let us recall a fact from measure theory.

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Fact from measure theory. Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a measurable map between to measurable spaces $\mathcal{X}$ and $\mathcal{Y}$, and let μ be a measure on $\mathcal{X}$. For every measurable $f : \mathcal{Y} \to \mathbb{R}$, we have the identity

$$\int f \circ \varphi \, \mathrm{d}\mu = \int f \, \mathrm{d}(\varphi\mu) \; .$$

To verify this, we can start with the case in which $f = \mathbb{1}_B$ is the indicator of a measurable set $B \subseteq \mathcal{Y}$. Note that $\mathbb{1}_B \circ \varphi = \mathbb{1}_{\varphi^{-1}B}$, hence

$$\int \mathbb{1}_B \circ \varphi \, \mathrm{d}\mu = \int \mathbb{1}_{\varphi^{-1}B} \, \mathrm{d}\mu = \mu(\varphi^{-1}B) = (\varphi\mu)(B) = \int \mathbb{1}_B \, \mathrm{d}(\varphi\mu)$$

by the definition of the measure $\varphi\mu$. The case in which f is a simple function is then covered easily. To verify the claim when f is a non-negative measurable function, we use approximation by simple functions. Lastly, to cover the case of an arbitrary integrable f, we decompose f into its positive and negative parts. $\diamond$

The following proof has a bug. See if you can spot it.

"Proof" of Kac's theorem. The first step is to make sure that τ_A^+ is a measurable map.

Exercise. Verify that $\tau_A^+ : \mathcal{X} \to \mathbb{R}$ is a measurable map.

To proceed, observe that for $x \in \widetilde{A}$,

$$\tau_A^+(x) = \begin{cases} 1 & \text{if } x \in T^{-1}A, \\ 1 + \tau_A^+\big(T(x)\big) & \text{if } x \in T^{-1}A^{\mathsf{c}}. \end{cases}$$

Hence,

$$\begin{aligned} \int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu &= \underbrace{\int_{\widetilde{A}} 1 \, \mathrm{d}\mu}_{\mu(\widetilde{A})} + \int_{\widetilde{A}} \mathbb{1}_{T^{-1}A^{\mathsf{c}}}(x) \cdot \tau_A^+(x) \mu(\mathrm{d}x) \\ &= \mu(\widetilde{A}) + \underbrace{\int \mathbb{1}_{\widetilde{A} \cap T^{-1}A^{\mathsf{c}}}(x) \cdot \tau_A^+(x) \mu(\mathrm{d}x)}_{\circledast} \; . \end{aligned} \tag{1}$$

Exercise. Verify that $\widetilde{A} \cap T^{-1}A^{\mathsf{c}} = T^{-1}(\widetilde{A} \setminus A) \uplus (A \setminus T^{-1}\widetilde{A})$.

By Poincaré's theorem, $\mu(A \setminus T^{-1}\widetilde{A}) = 0$. It follows that

$$\mathbb{1}_{\widetilde{A} \cap T^{-1}A^{\mathsf{c}}} = \mathbb{1}_{T^{-1}(\widetilde{A} \setminus A)} = \mathbb{1}_{\widetilde{A} \setminus A} \circ T \quad \mu\text{-a.e.} \; .$$

Hence,

$$\begin{aligned} \circledast &= \int (\mathbb{1}_{\widetilde{A} \setminus A} \cdot \tau_A^+) \circ T \, \mathrm{d}\mu \\ &= \int \mathbb{1}_{\widetilde{A} \setminus A} \cdot \tau_A^+ \, \mathrm{d}(T\mu) && \text{(by the above fact from measure theory)} \\ &= \int \mathbb{1}_{\widetilde{A} \setminus A} \cdot \tau_A^+ \, \mathrm{d}\mu && \text{(because } T \text{ preserves } \mu\text{)} \\ &= \int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu - \int_A \tau_A^+ \, \mathrm{d}\mu && \text{(because } A \subseteq \widetilde{A}\text{).} \end{aligned} \tag{2}$$

Putting (1) and (2) together, we obtain

$$\cancel{\int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu} = \mu(\widetilde{A}) + \cancel{\int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu} - \int_A \tau_A^+ \, \mathrm{d}\mu \; ,$$

from which we obtain

$$\int_A \tau_A^+ \, \mathrm{d}\mu = \mu(\widetilde{A}) \; .$$

$\square$

As you might have noticed, the above proof relies on the implicit (and unjustified) assumption that τ_A^+ is integrable over $\widetilde{A}$ (i.e., $\int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu < \infty$).

Exercise. Fix the bug in the above proof. *Hint*: Let M be a large number and repeat the reasoning for $\int_{\widetilde{A}} (\tau_A^+ \wedge M) \, \mathrm{d}\mu$ instead of $\int_{\widetilde{A}} \tau_A^+ \, \mathrm{d}\mu$. At the end, let $M \to \infty$. (Here, $a \wedge b$ stands for the minimum of a and b.)

Alternative proofs of Kac's theorem can be found in the book by Viana and Oliveira (Theorem 1.2.2) and the book by Einsiedler and Ward (Theorem 2.44).