Poincaré's recurrence theorem

The orbit of every invertible finite-state dynamical system returns to its initial state infinitely many times. This is simply because every orbit in such a system is periodic. Poincaré showed that a slightly weaker recurrence property holds for every measure-preserving dynamical system.

Theorem (Poincaré's Recurrence Theorem; 1890)**.** *Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$. Then, for μ-a.e. $x \in A$, we have $T^n(x) \in A$ for some $n \geq 1$. In fact, the orbit of μ-a.e. $x \in A$ returns to A infinitely many times.*

We prove the first assertion. Proof of the second assertion will be left as an exercise.

Proof of the first claim. Let E denote the set of all points in A whose orbits do not return to A, that is,

$$E \coloneqq \{x \in A : T^k(x) \notin A \text{ for all } k \in \mathbb{Z}_+\} .$$

Note that E is a measurable set because it can be written as $E = A \cap \left(\bigcap_{k=1}^{\infty} T^{-k} A^{\mathsf{c}}\right)$. We want to show that $\mu(E) = 0$.

We start by noting that

Claim. The sets $E, T^{-1}E, T^{-2}E, \ldots$ are disjoint.

Argument. Suppose $y \in T^{-i}E \cap T^{-j}E$ for some $0 \leq i < j$. Then, $x \coloneqq T^i(y) \in E \cap T^{-(j-i)}E$. Hence, $x \in E$ and $T^{(j-i)}(x) \in E \subseteq A$, where $j - i \in \mathbb{Z}_+$, contradicting the definition of E. Therefore, $T^{-i}E \cap T^{-j}E = \varnothing$ for all $0 \leq i < j$.

It follows that

$$\mu(E) + \mu(T^{-1}E) + \mu(T^{-2}E) + \cdots = \mu\Big(\bigcup_{k=0}^{\infty} T^{-k}E\Big) \leq \mu(\mathcal{X}) = 1 .$$

But since T preserves μ, we have

$$\mu(E) = \mu(T^{-1}E) = \mu(T^{-2}E) = \cdots .$$

Therefore, we must have $\mu(E) = 0$. □

Exercise. Prove the second assertion in Poincaré's theorem.

Hint: Either adapt the proof of the 1st assertion, or use the 1st assertion as a lemma.

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