

Von Neumann's Ergodic Theorem

Theorem (Von Neumann's Ergodic Theorem; 1932). *Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . For every $f \in L^2_\mu(\mathcal{X})$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow \tilde{f} \quad \text{in } L^2_\mu(\mathcal{X}) \text{ as } n \rightarrow \infty$$

for some $\tilde{f} \in L^2_\mu(\mathcal{X})$ that satisfies $\tilde{f} \circ T = \tilde{f}$ μ -almost everywhere.

Remark (Characterization of the limit). The proof below will characterize $\tilde{f}$ as the orthogonal projection of f onto the closed linear subspace

$$\{g \in L^2_\mu(\mathcal{X}) : g \circ T = g \text{ } \mu\text{-a.e.}\}$$

of $L^2_\mu(\mathcal{X})$. ◇

Remark (Induced isometry). The map $f \mapsto f \circ T$ is a positive linear isometry of $L^2_\mu(\mathcal{X})$. To see that it is an isometry (i.e., it preserves distances in L^2_μ), note that

$$\|f_2 \circ T - f_1 \circ T\|^2 = \int |f_2 \circ T - f_1 \circ T|^2 d\mu = \int |f_2 - f_1|^2 d(T\mu) = \int |f_2 - f_1|^2 d\mu = \|f_2 - f_1\|^2.$$

The linearity and the positivity ($f \geq 0$ implies $f \circ T \geq 0$) are clear. ◇

Proof of von Neumann's theorem. We start with some special cases:

- If $f \circ T = f$ μ -a.e., then the claim trivially holds and $\tilde{f} = f$.
- If $f = h \circ T - h$ for some $h \in L^2_\mu(\mathcal{X})$, then $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow 0$ in $L^2_\mu(\mathcal{X})$, hence $\tilde{f} = 0$.

Argument. Note that

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = \frac{1}{n} \sum_{k=0}^{n-1} (h \circ T - h) \circ T^k = \frac{1}{n} \sum_{k=0}^{n-1} (h \circ T^{k+1} - h \circ T^k) = \frac{h \circ T^n - h}{n}.$$

Therefore,

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right\| = \frac{1}{n} \|h \circ T^n - h\| \leq \frac{1}{n} (\|h \circ T^n\| + \|h\|) = \frac{2}{n} \|h\|,$$

which goes to 0 as $n \rightarrow \infty$. (For the last equality, we have used the fact that $h \mapsto h \circ T$ is an isometry.)

- If $f = c_1 f_1 + c_2 f_2$ for some $f_1, f_2 \in L^2_\mu(\mathcal{X})$ and $c_1, c_2 \in \mathbb{R}$, and the claim holds for f_1 and f_2 , then the claim clearly also holds for f with $\tilde{f} = c_1 \tilde{f}_1 + c_2 \tilde{f}_2$.

Let

$$B := \{h \circ T - h : h \in L^2_\mu(\mathcal{X})\}$$

and note that B is a linear subspace of $L^2_\mu(\mathcal{X})$. Let us consider one more special case:

Notes for Introduction to Ergodic Theory (MATH 307K), Siamak Taati
Last update: February 27, 2025

- If $f \in \overline{B}$, then $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow 0$ in $L_\mu^2(\mathcal{X})$, hence $\tilde{f} = 0$.

(Here, $\overline{B}$ stands for the closure of B in $L_\mu^2(\mathcal{X})$.)

□

Argument. The case in which $f \in B$ is already handled. Let $f \in \overline{B}$ and let $f_1, f_2, \dots \in B$ be such that $f_m \rightarrow f$ as $m \rightarrow \infty$.

Observe that for a fixed m ,

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} f_m \circ T^k + \frac{1}{n} \sum_{k=0}^{n-1} (f - f_m) \circ T^k \right\| \\ &\leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} f_m \circ T^k \right\| + \frac{1}{n} \sum_{k=0}^{n-1} \underbrace{\|(f - f_m) \circ T^k\|}_{\|f - f_m\|} \\ &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} f_m \circ T^k \right\| + \|f - f_m\|. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right\| \leq \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f_m \circ T^k \right\| + \|f - f_m\| = \|f - f_m\|.$$

Finally, letting $m \rightarrow \infty$, we find that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right\| = 0.$$

⊥

Now, let

$$I := \{g \in L_\mu^2(\mathcal{X}) : g \circ T = g \text{ } \mu\text{-a.e.}\},$$

and observe that I is a closed linear subspace of $L_\mu^2(\mathcal{X})$.

So far, we have verified that the conclusion of the theorem holds if f belongs to either of the two closed subspaces I and $\overline{B}$. Remarkably, every element of $L_\mu^2(\mathcal{X})$ can be decomposed (in a unique fashion) as a sum of two elements one belonging to I and the other belonging to $\overline{B}$. More specifically:

Lemma (Decomposition lemma). *Every $f \in L_\mu^2(\mathcal{X})$ has a unique decomposition $f = \tilde{f} + f_0$ where*

$$\tilde{f} \in I, \quad f_0 \in \overline{B}, \quad \tilde{f} \perp f_0.$$

In other words, $L_\mu^2(\mathcal{X}) = I \oplus \overline{B}$.

(The proof of the lemma comes below.)

It follows that for every $f \in L_\mu^2(\mathcal{X})$,

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} \tilde{f} \circ T^k}_{\tilde{f}} + \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} f_0 \circ T^k}_{\rightarrow 0} \rightarrow \tilde{f} \quad \text{in } L_\mu^2(\mathcal{X}) \text{ as } n \rightarrow \infty,$$

and this concludes the proof. □

Proof of the decomposition lemma. We need to show that I is the orthogonal complement of $\overline{B}$.

First, let $g \in I$. Then, for every $h \in L_\mu^2(\mathcal{X})$, we have

$$\begin{aligned} \int (h \circ T - h) \bar{g} \, d\mu &= \int (h \circ T) \bar{g} \, d\mu - \int h \bar{g} \, d\mu \\ &= \int (h \circ T) (\bar{g} \circ T) \, d\mu - \int h \bar{g} \, d\mu && \text{(because } g \circ T = g \text{ } \mu\text{-a.e.)} \\ &= \int h \bar{g} \, d(T\mu) - \int h \bar{g} \, d\mu \\ &= 0 && \text{(because } T\mu = \mu) \end{aligned}$$

hence $g \perp (h \circ T - h)$. It follows that $g \perp \overline{B}$, and in particular $I \subseteq \overline{B}^\perp$.

Conversely, let $g \in \overline{B}^\perp$. Then, for every $h \in L_\mu^2(\mathcal{X})$, we have $\int (h \circ T - h)\overline{g} \, d\mu = 0$, which implies

$$\int (h \circ T)\overline{g} \, d\mu = \int h\overline{g} \, d\mu. \quad (\clubsuit)$$

Now, we can write

$$\begin{aligned} \int |g \circ T - g|^2 \, d\mu &= \int (g \circ T - g)(\overline{g \circ T - g}) \, d\mu \\ &= \int (g \circ T)(\overline{g \circ T}) \, d\mu - \int (g \circ T)\overline{g} \, d\mu + \int g\overline{g} \, d\mu - \int g(\overline{g \circ T}) \, d\mu. \end{aligned}$$

Observe that all the four terms on the right-hand side are equal to $\int g\overline{g} \, d\mu$: the first term because of the invariance of μ , and the 2nd and the 4th by $(\clubsuit)$. Therefore, $\int |g \circ T - g|^2 \, d\mu = 0$, which implies $g \in I$. It follows that $\overline{B}^\perp \subseteq I$.

We conclude that $\overline{B}^\perp = I$. □

Remark (Extension to higher dimensions). The above proof can be readily generalized to higher dimensional dynamics, where the transformation T is replaced with a measure-preserving action of $\mathbb{Z}^d$ on $\mathcal{X}$. More generally, the same argument works if $\mathbb{Z}^d$ is replaced with any countable amenable group. ◇

Remark (Hilbert space variant). The theorem and its proof rely only on the fact that $L_\mu^2(\mathcal{X})$ is a Hilbert space and $f \mapsto f \circ T$ is a linear isometry.

The theorem extends to an abstract setting where $L_\mu^2(\mathcal{X})$ is replaced with a Hilbert space and $\mathcal{H}$ and $f \mapsto f \circ T$ is replaced with a linear contraction $U : \mathcal{H} \rightarrow \mathcal{H}$. ◇

Exercise. Formulate and prove the abstract version of von Neumann's theorem mentioned in the latter remark.