Mixing

Convergence in density

Natural density. The *upper* and the *lower density* of a set $J \subseteq \mathbb{N}$ are defined as

$$\overline{d}(J) := \limsup_{n\to\infty} \frac{|J \cap [0,n)|}{n} \;,$$
$$\underline{d}(J) := \liminf_{n\to\infty} \frac{|J \cap [0,n)|}{n} \;.$$

If the upper and the lower density of J coincide, we call the common value the *density* of J and denote it by $d(J)$.

For instance,

$$d(\varnothing) = 0 \;, \qquad d(\mathbb{N}) = 1 \;, \qquad d\big(\{2k : k \in \mathbb{N}\}\big) = d\big(\{2k+1 : k \in \mathbb{N}\}\big) = 1/2 \;.$$

The following is an example of a set which does not have a well-defined density.

Example (A set without density)**.** Each $J \subseteq \mathbb{N}$ can naturally be represented by its indicator sequence $x \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ where $x_i = \mathtt{1}$ if and only if $i \in J$. Conversely, every sequence $x \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ is the indicator sequence of one and only one set $J \subseteq \mathbb{N}$.

Consider the set J whose indicator sequence has the following form:

$$x_i := \begin{cases} \mathtt{1} & \text{if } N_{2k} \leq i < N_{2k+1} \text{ for some } k \in \mathbb{N}, \\ \mathtt{0} & \text{otherwise.} \end{cases}$$

where $0 = N_0 < N_1 < N_2 < N_3 < \cdots$ are such that $N_k/N_{k+1} \to 0$ as $k \to \infty$. (For instance, we can choose $N_k := 2^{k^2}$.) Then, it is easy to verify that $\underline{d}(J) = 0 \neq 1 = \overline{d}(J)$. ○

We say that $J \subseteq \mathbb{N}$ has *full density* if $d(J) = 1$.

Remark. The natural density is somewhat like a probability measure on $\mathbb{N}$:

- (additivity) If $J_1 \cap J_2 = \varnothing$, then $d(J_1 \cup J_2) = d(J_1) + d(J_2)$ provided $d(J_1)$ and $d(J_2)$ exist.
- (monotonicity) If $J_1 \subseteq J_2$, then $d(J_1) \leq d(J_2)$ provided $d(J_1)$ and $d(J_2)$ exist.
- (sub-additivity) $d(J_1 \cup J_2) \leq d(J_1) + d(J_2)$ provided all three densities exist.

However, it is not countably additive, and lacks continuity:

- If $J_n := \{n\}$, then $d\big(\bigcup_{n=0}^{\infty} J_n\big) = 1 \neq 0 = \sum_{n=0}^{\infty} d(J_n)$.
- If $J_n := \{n, n+1, n+2, \ldots\}$, then $d\big(\bigcap_{n=0}^{\infty} J_n\big) = 0 \neq 1 = \lim_{n\to\infty} d(J_n)$. ◇

Full-density diagonal trick. Recall the *diagonal* trick often used in analysis: if

$$\big(a_n^{(0)}\big)_{n\in\mathbb{N}}, \big(a_n^{(1)}\big)_{n\in\mathbb{N}}, \big(a_n^{(2)}\big)_{n\in\mathbb{N}}, \ldots$$

is a sequence of sequences, each being a subsequence of the previous one, then there exists a sequence (the "diagonal" sequence) which is eventually a subsequence of each $\big(a_n^{(k)}\big)_{n\in\mathbb{N}}$.

Clearly the diagonal trick is about the indices rather than the values of the sequences. The following is a concise formulation of the trick:

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Lemma (Diagonal sequence)**.** *Let $\mathbb{N} = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ be a sequence of infinite sets. There exists an infinite set $J \subseteq \mathbb{N}$ (the "diagonal" set) such that $J \setminus J_k$ is finite for each $k \in \mathbb{N}$.*

Here, we should think of J_k as the set of indices of the k-th subsequence of $\left(a_n^{(0)}\right)_{n\in\mathbb{N}}$.

It turns out that if all the sets J_k have full density, then the diagonal set can also be chosen to have full density.

Lemma (Full-density diagonal sequence)**.** *Let $\mathbb{N} = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ be a sequence of sets such that $d(J_k) = 1$ for each $k \in \mathbb{N}$. There exists a set $J \subseteq \mathbb{N}$ (the "diagonal" set) such that* (i) *$J \setminus J_k$ is finite for each $k \in \mathbb{N}$, and* (ii) *$d(J) = 1$.*

Proof. Pick $0 = N_0 < N_1 < N_2 < N_3 < \cdots$ such that for each k, we have

$$\frac{|J_k \cap [0,n)|}{n} > 1 - 2^{-k} \qquad \text{for all } n \geq N_k.$$

Set

$$J \coloneqq \bigcup_{k=0}^{\infty} \Big(J_k \cap [N_k, N_{k+1}) \Big) \ .$$

Then,

(i) For each k, we have $J \setminus J_k \subseteq [0, N_k)$ because $J_k \supseteq J_{k+1} \supseteq \cdots$. Hence, $J \setminus J_k$ is finite.

(ii) Note that if $n < N_{k+1}$, then $J \cap [0,n) \supseteq J_k \cap [0,n)$ because $J_k \subseteq J_{k-1} \subseteq \cdots \subseteq J_0$. For each n, let k_n denote the unique value such that $N_{k_n} \leq n < N_{k_n+1}$. Then,

$$\frac{|J \cap [0,n)|}{n} \geq \frac{|J_{k_n} \cap [0,n)|}{n} > 1 - 2^{-k_n} \ .$$

Since $k_n \to \infty$ as $n \to \infty$, we find that

$$\underline{d}(J) = \liminf_{n\to\infty} \frac{|J \cap [0,n)|}{n} \geq \liminf_{n\to\infty} \left(1 - 2^{-k_n}\right) = 1 \ .$$

Therefore, $d(J) = 1$. □