American University of Beirut
STAT 210: Elementary Statistics for Sciences
2022–2023 Fall

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Chapter 10
Estimation and hypothesis testing about two populations
Part 3

Comparing two populations

Comparing two populations

Inference about population means

We consider the following scenarios:

1. independent samples, σ_1 and σ_2 known,
   (a) normal populations
   (b) large samples
2. independent samples, σ_1 and σ_2 unknown but $\sigma_1 = \sigma_2$,
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3. independent samples, σ_1 and σ_2 unknown and $\sigma_1 \neq \sigma_2$,
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4. paired sample,
   (a) normal population of pair differences
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Inference about population proportions

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Comparing two means: paired samples

Example (Chewing tobacco and heart rate)

[This example is taken from S. Ross, Introductory Statistics, Elsevier, 2010]

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Q What are the competing hypotheses?

$\boxed{\text{A}}$ H_0 $\mu_d = 0$ (tobacco does not increase heart rate)

H_1 $\mu_d > 0$ (tobacco increases heart rate)

where μ_d is the mean difference in heart rate of the population of adults before and after chewing tobacco for one minute.

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$$\boxed{H_0:\ \mu_d = 0 \qquad H_1:\ \mu_d > 0}$$

To test this, they conduct an experiment involving 12 subjects. Each subject is asked to chew tobacco for one minute. The researchers measure the heart rate of the subjects before and after chewing tobacco and calculate the difference and compare them.

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Q What is a suitable test statistic?

Comparing two means: paired samples

Example (Chewing tobacco and heart rate)

The available statistical evidence is of the form

$$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots, (X_{12}, Y_{12})$$

where

- X_1 and Y_1 are the heart rates of the 1st subject before and after chewing tobacco.
- X_2 and Y_2 are the heart rates of the 2nd subject before and after chewing tobacco.
- ...
- X_{12} and Y_{12} are the heart rates of the 12th subject before and after chewing tobacco.

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The available statistical evidence is of the form

$$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots, (X_{12}, Y_{12})$$

Let

- $D_1 := Y_1 - X_1$ be the difference in the heart rate of the 1st subject before and after chewing tobacco.
- $D_2 := Y_2 - X_2$ be the difference in the heart rate of the 2nd subject before and after chewing tobacco.
- ...
- $D_{12} := Y_{12} - X_{12}$ be the difference in the heart rate of the 12th subject before and after chewing tobacco.

Comparing two means: paired samples

Example (Chewing tobacco and heart rate)

The available statistical evidence is of the form

$$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots, (X_{12}, Y_{12})$$

The relevant information is in the pair differences

$$D_1, D_2, D_3, \ldots, D_{12}$$

Comparing two means: paired samples

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Note that the mean

$$\overline{D} := \frac{D_1 + D_2 + D_3 + \cdots + D_{12}}{12}$$

is a point estimator for the population mean μ_d of the pair differences,

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Hence, we can treat the problem as if we have a single population and a single sample of size 12 from that population.

Comparing two means: paired samples

Example (Chewing tobacco and heart rate)

$$\boxed{\text{H}_0\text{: } \mu_d = 0 \qquad \text{H}_1\text{: } \mu_d > 0}$$

Q What is a suitable test statistic?

Comparing two means: paired samples

Example (Chewing tobacco and heart rate)

$$\boxed{H_0\text{: } \mu_d = 0 \qquad H_1\text{: } \mu_d > 0}$$

It is fair to assume that, within the population, the pair difference in heart rate before and after chewing tobacco is (approximately) normally distributed.

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A The sample mean of the pair differences is also normally distributed with mean μ_d and standard deviation $\sigma_d/\sqrt{12}$, where σ_d is the population standard deviation of the pair differences.

But we do not have access to its standard deviation, because we do not know σ_d of the pair differences.

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A Instead, we can use

$$T := \frac{\overline{D} - \mu_d}{S_d/\sqrt{12}}$$

where S_d is the sample standard deviation of the pair differences.

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A t distribution with $df = 12 - 1 = 11$.

Comparing two means: paired samples

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Assumption: normal populations of pair differences

subjects	1	2	3	...	12	mean	sd
before	73	67	68	...	78	71.5	7.73
after	77	69	73	...	80	75.25	7.55
change	4	2	5	...	2	3.75	3.08

The above table shows the result of the experiment (in beats per minute).

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$$t =$$

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A The observed value of the test statistic is

$$t = \frac{3.75 - 0}{3.08/\sqrt{12}} \approx$$

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Since p-value $\leq \alpha$, the researchers should reject the null hypothesis.

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Conclusion: The evidence is statistically significant (with p-value $< 0.1\%$) in favor of the claim that chewing tobacco increases the heart rate.

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$$\begin{aligned} 98\%\text{-CI} &= \bar{d} \pm t \times \frac{s_d}{\sqrt{12}} \\ &= 3.75 \pm 2.71808 \times \frac{3.08}{\sqrt{12}} \\ &\approx \boxed{3.75 \pm 2.4167\,\text{bpm}} \end{aligned}$$

Comparing two means: paired samples

Covered scenarios:

I. The samples are large.

II. Within the paired population,
the differences are (approximately) normal.

We can do inference about the population mean of pair differences in the same manner as we do about the mean of a single population. (View the pairs as the “population” and the pair differences as the variable of interest.)

Comparing two populations

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Comparing two proportions: indep. large samples

We would like to compare the proportion of entities with a certain characteristic $\circledast$ (e.g., having blue feather) in two populations (e.g., populations of male and female birds within a species).

Competing hypotheses [for a one-sided test]

H$_0$ $p_2 = p_1$ (the proportion is the same in both populations)

H$_0$ $p_2 > p_1$ (the proportion in population #2 is higher)

Evidence

- A large sample from population #1.
- A large sample from population #2.
- The two samples are independent.

Comparing two proportions: indep. large samples

We would like to compare the proportion of entities with a certain characteristic $\circledast$ (e.g., having blue feather) in two populations (e.g., populations of male and female birds within a species).

Competing hypotheses

[for a one-sided test]

H_0 $p_2 = p_1 + a$ (the two proportions differ by a)

H_0 $p_2 > p_1 + a$ (the two proportions differ by more than a)

Evidence

- A large sample from population $\#1$.
- A large sample from population $\#2$.
- The two samples are independent.

Comparing two proportions: indep. large samples

$$\boxed{\text{H}_0\text{: } p_2 - p_1 = a \qquad \text{H}_1\text{: } p_2 - p_1 > a}$$

Sample proportions

The sample proportions are

$$\widehat{p}_1 := \frac{m_1}{n_1} \qquad \text{and} \qquad \widehat{p}_2 := \frac{m_2}{n_2}$$

where n_1 and n_2 are the sample sizes and m_1 and m_2 are the number of entities with characteristic $\circledast$ in each sample.

The pooled sample proportion is obtained by combining the two samples:

$$\bar{p} := \frac{m_1 + m_2}{n_1 + n_2} = \frac{n_1\widehat{p}_1 + n_2\widehat{p}_2}{n_1 + n_2} \; .$$

Comparing two proportions: indep. large samples

$$\boxed{\text{H}_0\text{: } p_2 - p_1 = a \qquad \text{H}_1\text{: } p_2 - p_1 > a}$$

Choice of statistic

We can use the difference $\widehat{p}_2 - \widehat{p}_1$ as the test statistic.

Assuming the samples are large and independent, the distribution of $\widehat{p}_2 - \widehat{p}_1$ according to H$_0$ is approximately normal with

$$\text{mean} = a \qquad \text{stand. dev.} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \ .$$

Since H$_0$ does not specify the values of p_1 and p_2 individually, the standard deviation of $\widehat{p}_2 - \widehat{p}_1$ is unknown. Instead, we use

$$SE := \sqrt{\overline{p}(1-\overline{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \ .$$

which is an estimate for the true standard deviation.