American University of Beirut

STAT 210: Elementary Statistics for Sciences

2022–2023 Fall

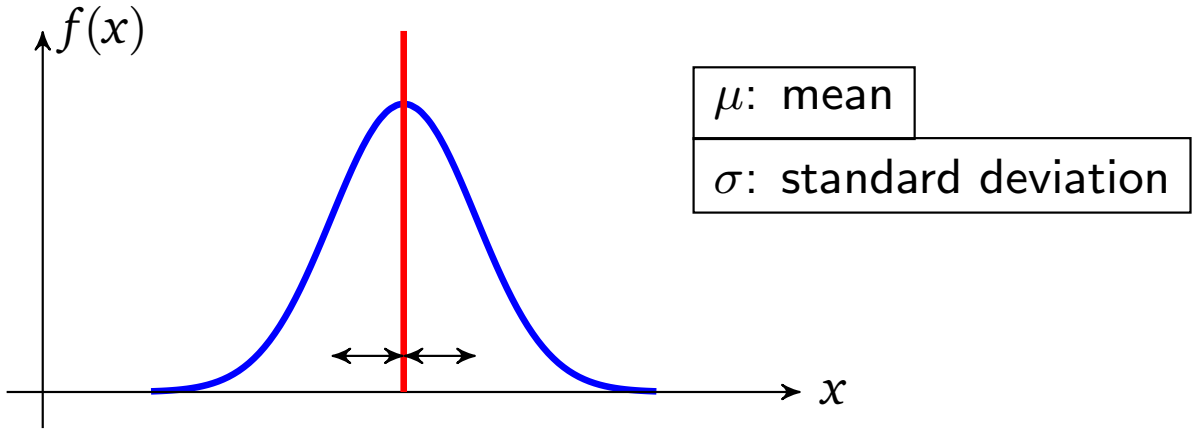
Siamak Taati

Chapter 6
Continuous random variables
Part 2

Continuous random variables

Normal RVs

The normal distribution with parameters μ and σ is a specific bell-shaped probability density function:



$$f(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

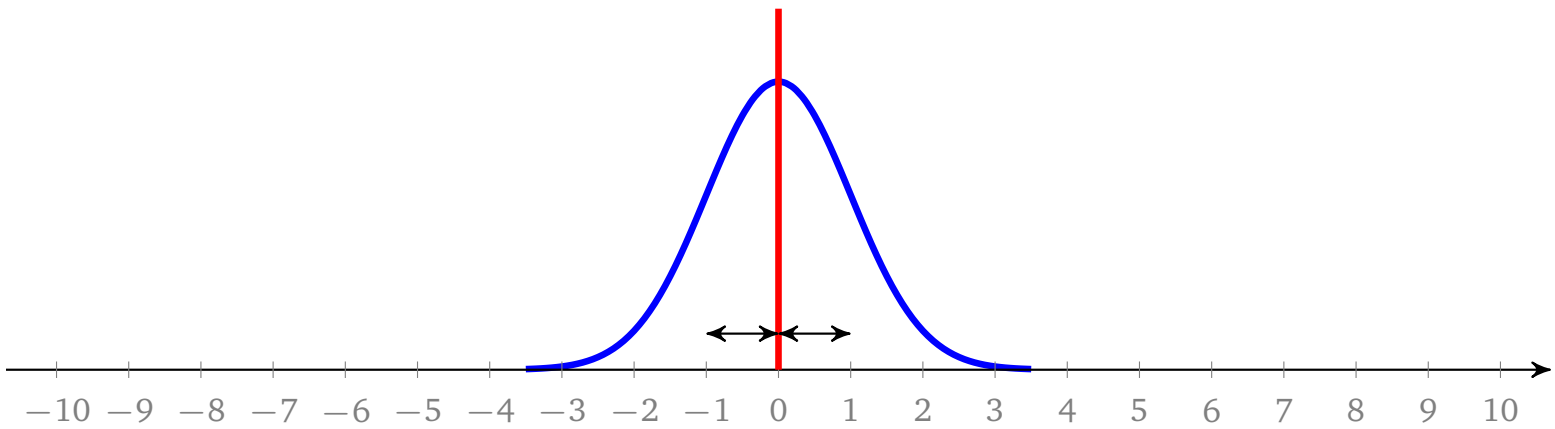
A continuous RV with a normal distribution is called a normal RV.

The normal distribution is symmetric and unimodal. Its right and left tails extend indefinitely, which is to say:

The possible values of a normal RV are all numbers
$$-\infty < x < +\infty.$$

Normal RVs

Let Z be a normal RV with mean 0 and standard deviation 1.

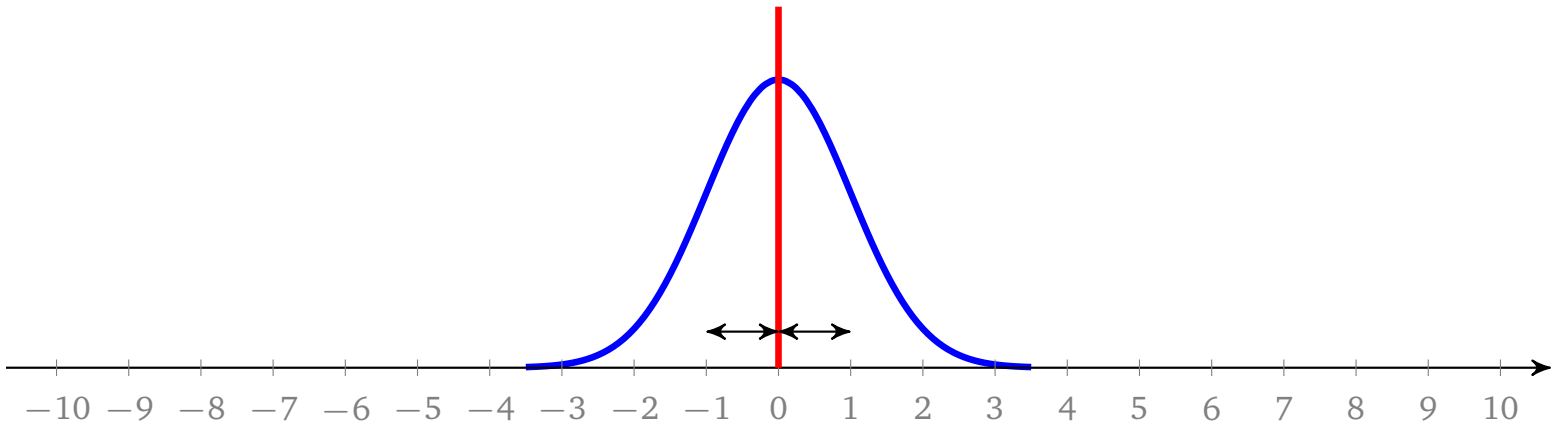


Probability density function of Z

Normal RVs

Let Z be a normal RV with mean 0 and standard deviation 1.

Q What is the distribution of $X := 2Z + 3$?



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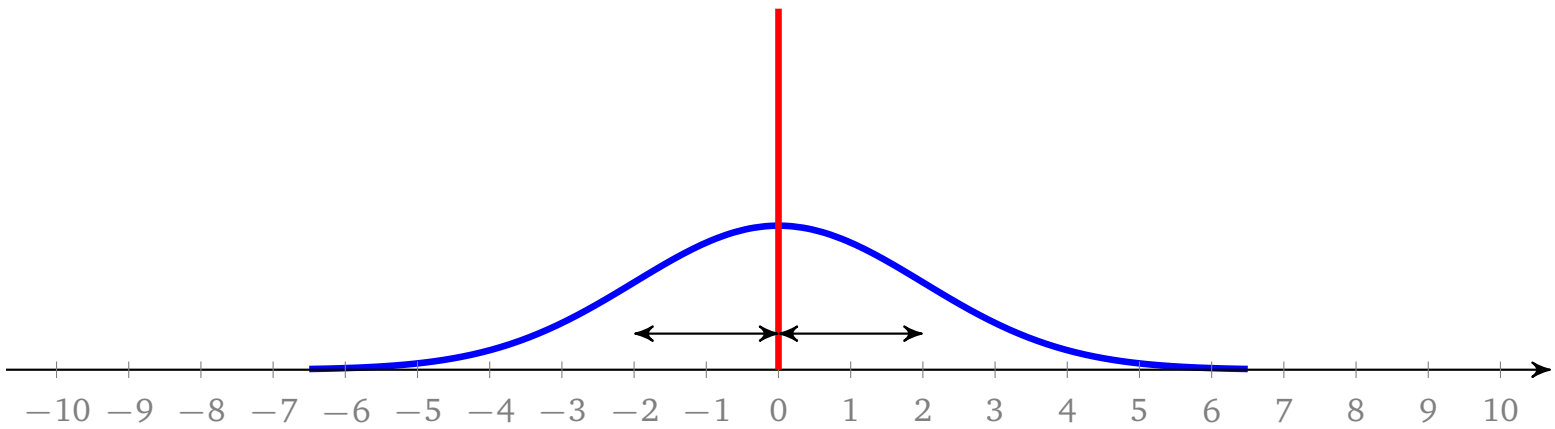

Probability density function of $2Z$

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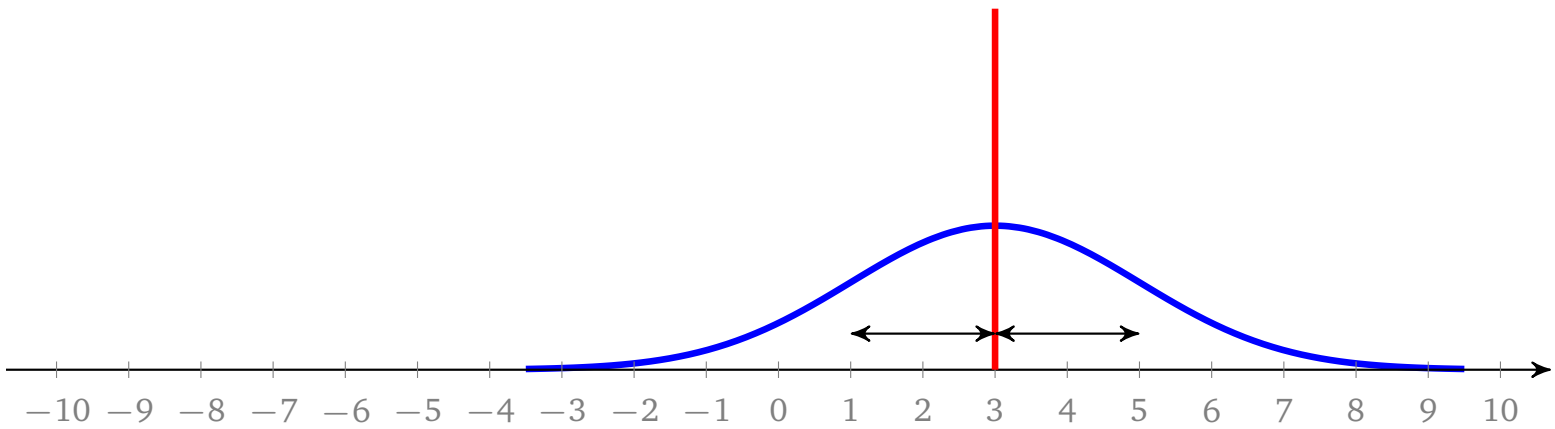

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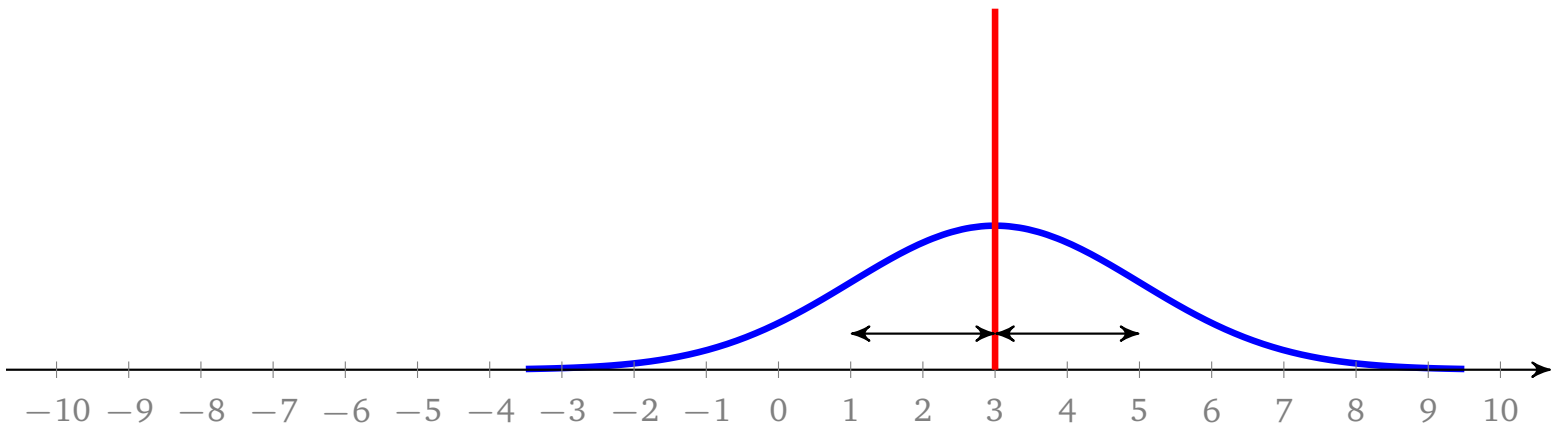

Probability density function of $2Z + 3$

A Normal with mean 3 and standard deviation 2.

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Scaling and shifting: normal distribution

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- $Y_1 := 2X + 3$ is also a normal RV.
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The parameters of the new RVs can be found using the following general principle:

Scaling and shifting: mean and standard deviation

If X is any (not necessarily normal) RV and a, b are arbitrary numbers, then

$$\boxed{\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b} \qquad \boxed{\mathbb{SD}[aX + b] = |a| \cdot \mathbb{SD}[X]}$$

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The height of 18-year-old individuals in Peru is approximately normally distributed.

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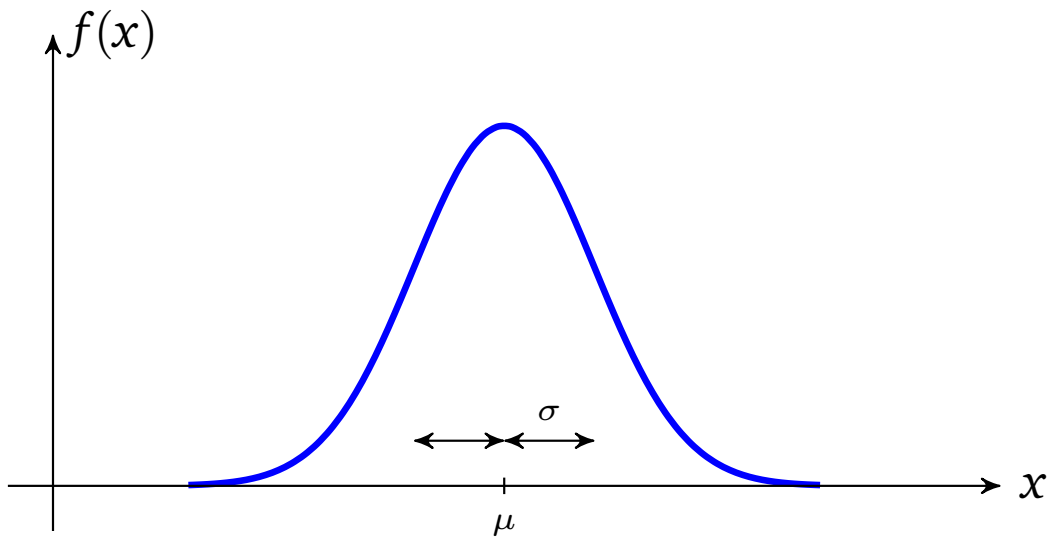

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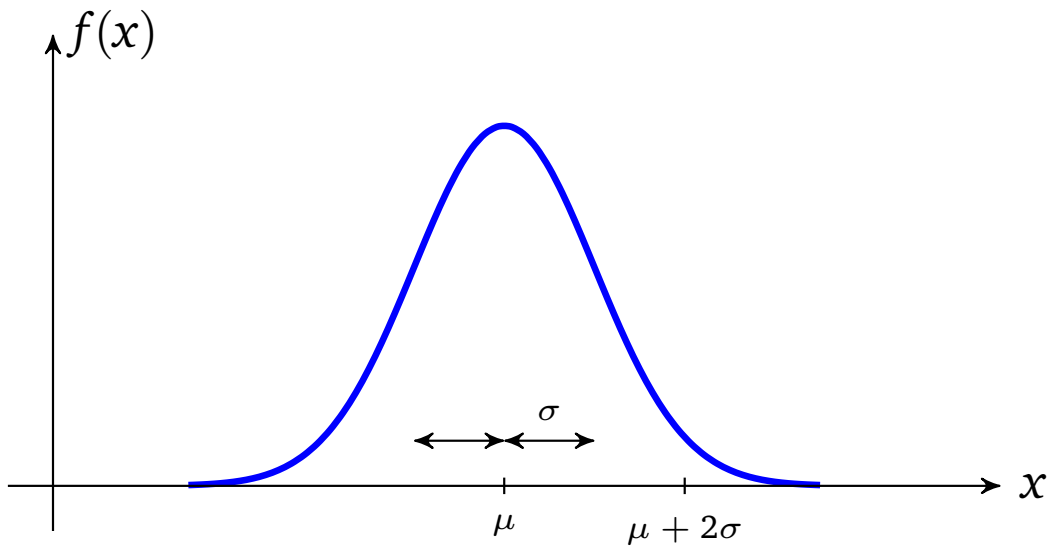

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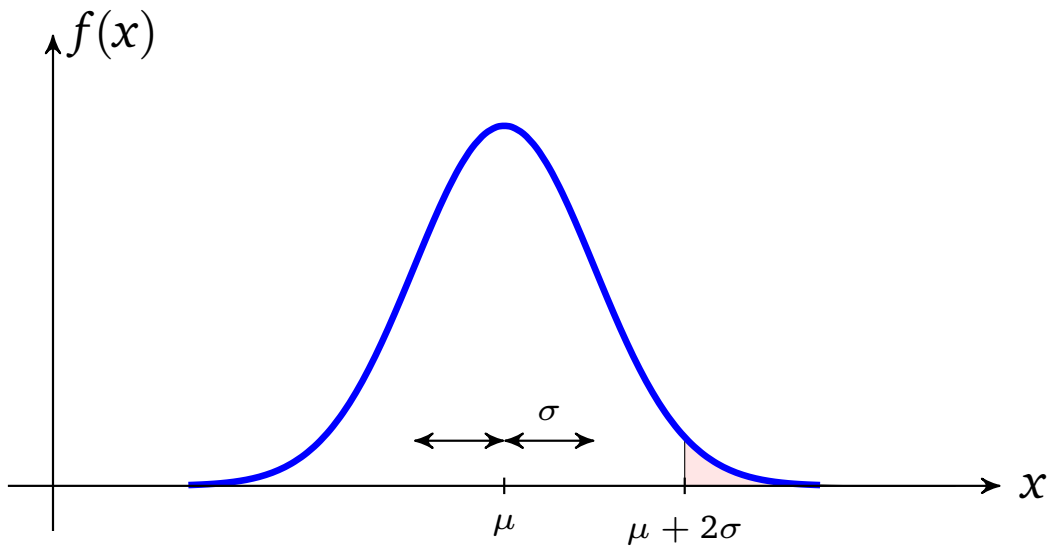

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$$\mathbb{P}(X > \mu + 2\sigma)$$

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Therefore, using the applet of normal distribution:

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Therefore, using the applet of normal distribution:

$$\mathbb{P}(X > \mu + 2\sigma) = \mathbb{P}(Z > 2) \approx \boxed{0.02275} \approx 2.3\% \ .$$

Standard normal RVs

A normal RV with mean 0 and standard deviation 1 is called a standard normal RV, and its distribution the standard normal distribution.

If X is a normal RV with mean μ and standard deviation σ, then

$$\boxed{Z := \frac{X - \mu}{\sigma}}$$

is a standard normal RV. It is called the standardized version of X.

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$$\mathbb{P}\left(\mu - 2\sigma < X < \mu + 2\sigma\right)$$

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$$\mathbb{P}\left(\mu - 2\sigma < X < \mu + 2\sigma\right) = \mathbb{P}(-2 < (X - \mu)/\sigma < 2)$$

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Using the applet of normal distribution:

$$\begin{aligned}\mathbb{P}(-2 < Z < 2) &= \mathbb{P}(Z < 2) - \mathbb{P}(Z < -2) \\ &\approx 0.97725 - 0.02275 = \boxed{0.9545} \approx 95\%\,.\end{aligned}$$

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[A] By symmetry, the mean of X is

$$\mu = (174 + 158)/2 = \boxed{166\,\text{cm}}$$

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$$0.9 = \mathbb{P}\left(158 < X < 174\right)$$

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Using the applet of normal distribution: $8/\sigma \approx 1.64485$.

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Using the applet of normal distribution: $8/\sigma \approx 1.64485$.

Therefore, $\sigma \approx \boxed{4.864\,\text{cm}}$