American University of Beirut

STAT 210: Elementary Statistics for Sciences

2022–2023 Fall

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Chapter 7

Sampling distributions

Part 1

Distribution of sample statistics

Samples in the language of probabilities

We earlier talked about samples as typical sources of data sets. We use samples as statistical evidence to learn about populations.

The sample statistics (e.g., sample mean, sample variance, ...) vary from sample to sample, and typically deviate from the corresponding population values (e.g., population mean, population variance, ...).

If the sample is taken at random, then the sample statistics will also be random. Armed with the language of probabilities, we can now view sample statistics as random variables and wonder about their distributions.

Samples in the language of probabilities

Random samples

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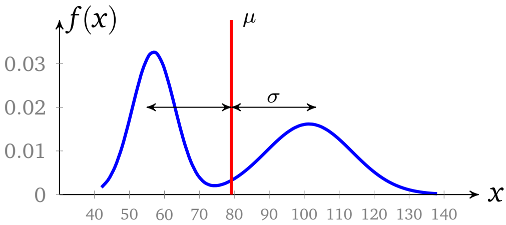
Sample mean

Example (Weight of orangutans)

The weights of adult orangutans are approximately distributed according to following probability density function, with mean $\mu = 79.15\,\text{Kg}$ and standard deviation $\sigma = 24.28\,\text{Kg}$.



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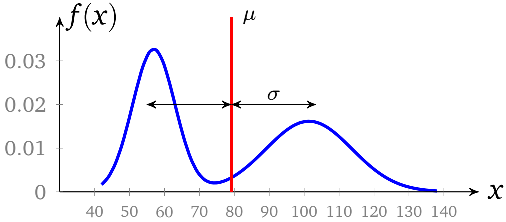

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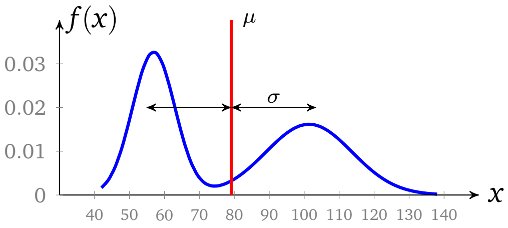
A The two peaks correspond to the typical weights of males and females.



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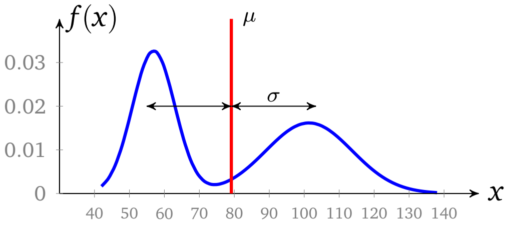

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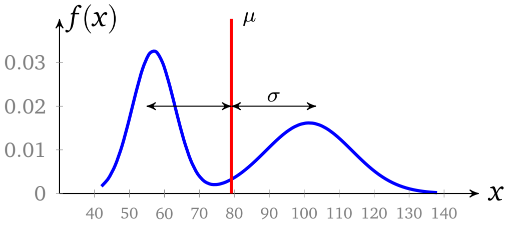

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A Taking a random sample of 50 orangutans is a random experiment. The weights of the picked orangutans

$$W_1, W_2, W_3, \ldots, W_{50}$$

are random variables in this experiment.

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are random variables in this experiment. Hence, the sample mean

$$\overline{W} := \frac{W_1 + W_2 + W_3 + \cdots + W_{50}}{50}$$

is again a random variable.

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be the sample mean. This is a random variable.

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In order to answer these questions, we first need to address the following questions:

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- Q What is the mean of $\overline{W}$?
- Q What is the standard deviation of $\overline{W}$?
- Q Can we determine the distribution of $\overline{W}$?

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Q What do we know about $W_1, W_2, W_3, \ldots, W_{50}$?

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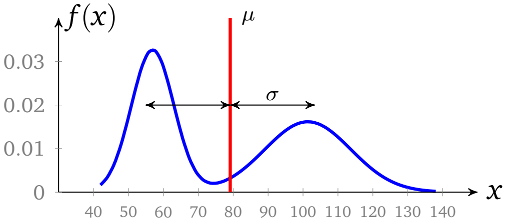
A ① Each of $W_1, W_2, W_3, \ldots, W_{50}$ is distributed according to the population distribution.



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1. Each of $W_1, W_2, W_3, \ldots, W_{50}$ is distributed according to the population distribution.
2. If the sample is taken with replacement, then $W_1, W_2, W_3, \ldots, W_{50}$ are independent.

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In general:

If the sample size is "small" compared to the population size, then the distinction between sampling with or without replacement becomes negligible! (Rule of thumb: $\frac{\text{sample size}}{\text{population size}} \leq 5\%$)

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Q What is the mean of $\overline{W}$?

A The mean of each W_i is the population mean μ. Hence,

$$\mathbb{E}[\overline{W}] = \mu = \boxed{79.15\,\mathrm{Kg}}$$

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A The standard deviation of each W_i is the population standard deviation σ. Assuming the sample is with replacement,

$$\mathbb{SD}[\overline{W}] = \frac{\sigma}{\sqrt{50}} = \frac{24.28}{\sqrt{50}} \approx \boxed{3.43\,\text{Kg}}$$

Sample mean

Mean and standard deviation of sample mean

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population with mean μ and standard deviation σ, then the sample mean

$$\overline{X} := \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is a random variable with

- $\boxed{\mathbb{E}[\overline{X}] = \mu}$ (valid for both types of sample)
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These are consequences of the following general principles:

If X and Y are arbitrary random variables, then

- $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
- $\mathbb{V}\mathrm{ar}[X+Y] = \mathbb{V}\mathrm{ar}[X] + \mathbb{V}\mathrm{ar}[Y]$ provided that X and Y are independent.

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Mean and standard deviation of sample mean: interpretation

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Mean and standard deviation of sample mean: interpretation

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- The fact that $\mathbb{E}[\overline{X}] = \mu$ means that

  $\overline{X}$ is an unbiased estimator for μ.

- The fact that $\mathbb{SD}[\overline{X}] = \sigma/\sqrt{n}$ reflects out expectation that

  The larger the sample, the more accurate the estimator.

  (The standard deviation of the sample mean is inversely proportional to the square root of the sample size.)