
Full Name: _____

Student No: _____

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 10 questions, some with multiple parts. All answers require justifications.
- During the exam, you are allowed to use your phone to access the app for probability distributions (and possibly a calculator app), but the internet access of your phone must be switched off.
- The duration of the exam is 2 hours.

You can use the remainder of this page as scratch paper.

1. (5 points)

(a) State the law of large numbers as precisely as you can, and give an example illustrating it.

(b) Is your answer to part (a) correct?

2. (5 points)

(a) State the central limit theorem as precisely as you can, and give an example illustrating it.

(b) Is your answer to part (a) correct?

3. (18 points) Determine which of the following statements is True and which is False. In each case, give a short justification.

_____ Saying that the grade of a student is in the upper 25th percentile of her class means that her grade is higher than or equal to at least $\frac{3}{4}$ of her classmates.

_____ If A and B are two events in a probability model, then necessarily $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$.

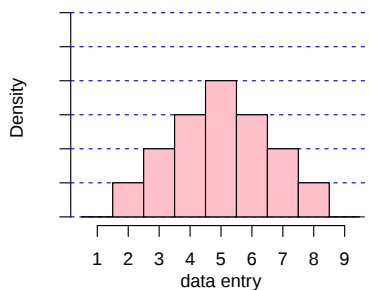
_____ If A and B are two disjoint (= mutually exclusive) events in a probability model, then necessarily $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$.

_____ The probability density function of every continuous random variable is a continuous function.

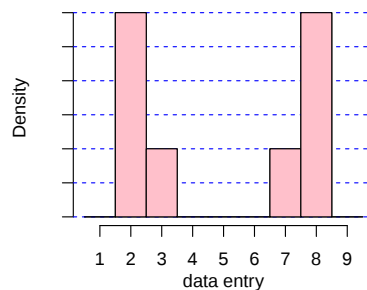
_____ If X and Y are two normal random variables, then the distribution of $X + Y$ is necessarily normal.

_____ If X and Y are two independent random variables with $X \sim \text{NBin}(r = 3, p = 0.7)$ and $Y \sim \text{NBin}(r = 5, p = 0.7)$, then $X + Y \sim \text{NBin}(r = 8, p = 0.7)$.

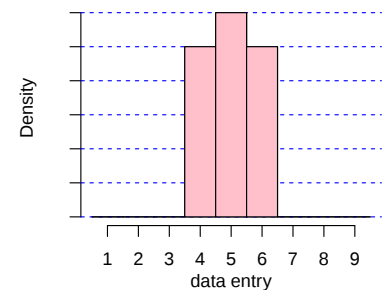
4. (5 points) Identify which of the following histograms has the largest and which has the smallest standard deviation.



(i)



(ii)



(iii)

Give a short justification.

5. (5 points) A random sample of 50 female orangutans is used to calculate the 95% confidence interval (33 Kg, 41 Kg) for the mean weight of female orangutans in Brunei. Which of the following statements is a correct interpretation of the latter statement?
- Of the entire population of female orangutans in Brunei, 95% weigh between 33 Kg and 41 Kg.
 - The chance that the mean weight of female orangutans in Brunei is between 33 Kg and 41 Kg is 95%.
 - The chance that a randomly picked female orangutan from Brunei weighs between 33 Kg and 41 Kg is 95%.
 - If we were to repeat taking samples from the female orangutans in Brunei and calculate a 95% confidence interval based on each sample, then about 95% of these intervals would contain the true population mean.

Justify!

6. (5 points) A coin has probability $p \neq 1/2$ of coming up heads. We flip the coin twice. Which of the following two events is more probable?

$A := \langle \text{the same face comes up in both flips} \rangle$

$B := \langle \text{difference faces come up} \rangle$

7. (8 points) Among a sample of 4000 newborns, 2080 turned out to be male. Find a 99% confidence interval for the probability p that a newborn is male.

8. (10 points) Let X and Y be two independent, standard normal random variables.

(a) Order the following probabilities from smallest to largest:

$$\mathbb{P}(X + Y > 1)$$

$$\mathbb{P}(X - Y > 1)$$

$$\mathbb{P}(2X > 1)$$

(b) Find $\text{Cov}[2X + Y + 1, X - 3Y - 2]$.

9. (20 points) An elevator can safely carry a load of up to 800 Kg. The weight of an individual can be modeled as a normal random variable with mean 75 Kg and standard deviation 10 Kg. We are wondering about ways to ensure that the safety cap on the load is complied with. As first glance, we suspect that loading the elevator with 10 passengers should be relatively safe, because that would allow roughly 80 Kg per person.

(a) What is the probability that a single passenger weighs more than 80 Kg?

(b) What is the probability that of 10 passengers, at least one weighs more than 80 Kg?

(c) What is the probability that 10 passengers together weigh more than the safe limit?

- (d) Explain why the probability calculated in part (c) is smaller than the probability calculated in part (b)
- (e) What is the maximum number of passengers in order for the probability of overloading to be at most 0.1%?

10. (20+ points) A *Geiger-Müller counter* (GM counter for short) is an instrument used for detecting ionized radiations. In normal circumstances, a GM counter records background radiation at an average rate of 1 tick per minute. While testing a GM counter in the backyard of your house, you notice occasional sudden surges of ticks followed by periods of silence. To understand this phenomenon, you decide to model the timing of the ticks on the GM counter with a Poisson process.

(a) What is the probability that within a 1-minute interval, the GM counter ticks more than 5 times?

(b) Divide a 30-minute interval into 30 1-minute sub-intervals. What is the probability that during at least one of these sub-intervals, the GM counter ticks more than 5 times?

(bonus) (c) Use the result of the previous part to argue that the chance that within 30 minutes, there is a 1-minute interval during which more than 10 ticks are recorded is less than 2%.

Still wondering about the observed occasional surges of ticks, you decide to test the hypothesis that the amount of radiations in your backyard is higher than ordinary. You start counting the ticks and notice that the 50th tick arrives in less than 30 minutes.

(d) Write down the precise statements of the two competing hypotheses.

(e) Compute the p-value.

(f) At a significance level of 1%, how should you conclude?

You can use this sheet as extra space for your solutions.

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