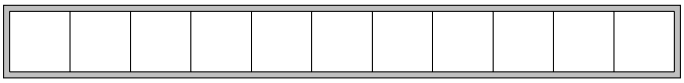
Full Name: ____________________

Student No: ____________________

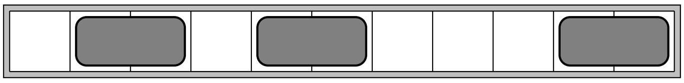
Grade:

For each answer, **provide a short argument**.

1. (4 points) We have a *board* consisting of a row of 11 unit squares:



We would like to arrange 3 *dominoes* (i.e., 2×1 rectangular blocks) on the board in such a way that every domino covers exactly two squares and no two dominoes overlap. An example of a valid arrangement of the dominoes is the following:

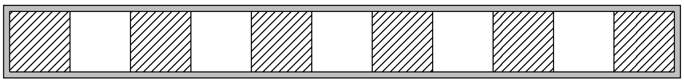


(a) How many valid arrangements are there?

Answer: $\binom{8}{3} = 56$

Solution: Every valid arrangement can be encoded by an 8-character string with 3 `D`'s (representing the dominoes) and 5 `H`'s (representing the "holes", or uncovered squares). Conversely, every such string encodes a valid arrangement of the dominoes. There are $\binom{8}{3} = 56$ such strings.

Suppose that every other square on the board is painted black, starting with the one on the left:



(b) What is the chance that, in a random valid arrangement of the dominoes, the left half of each domino sits on a black square?

Answer: $5/28 \approx 0.1786$

Solution: The sample space, Ω, is the set of all valid arrangements. Let E denote the event that the left half of each domino sits on a black square. Since all the possible outcomes are equally likely, we have

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}\ .$$

From part (a), we know that $|\Omega| = \binom{8}{3} = 56$. To count the elements of E, note that (i) each arrangement in E is uniquely identified by the position of the black squares it covers, and (ii) in each arrangement in E, the rightmost black square on the board is uncovered. Thus, counting the elements of E amounts to counting the number of ways one can choose 3 out of 5 black squares. Hence, $|E| = \binom{5}{3} = 10$, and as a result $\mathbb{P}(E) = 10/56 = 5/28$.

2. (4+ points) A jar contains 3 blue balls and 7 red balls. We draw a ball from the jar at random, record its color, and replace it back into the jar along with 2 more balls of the same color. Then, we draw another ball from the jar at random.

   (a) What is the chance that the second ball is blue?

   Answer: $3/10$

   Solution: The sample space is $\Omega = \{BB, BR, RB, RR\}$, where BB stands for the outcome that both drawn balls are blue, BR stands for the outcome that the first drawn ball is blue and the second is red, and so on. Let $B*$ denote the event that the first drawn ball is blue, and $*B$ denote the event that the second drawn ball is blue. The events $R*$ and $*R$ are defined similarly.
   Using the principle of total probability and the chain rule, we can write

$$\begin{aligned}\mathbb{P}(*B) &= \mathbb{P}(BB) + \mathbb{P}(RB) && \text{(principle of total probability)}\\ &= \mathbb{P}(B*)\,\mathbb{P}(*B \mid B*) + \mathbb{P}(R*)\,\mathbb{P}(*B \mid R*) && \text{(chain rule)}\\ &= \frac{3}{10}\cdot\frac{5}{12} + \frac{7}{10}\cdot\frac{3}{12}\\ &= \frac{3}{10}\cdot\left(\frac{5}{12}+\frac{7}{12}\right) = \frac{3}{10}\,.\end{aligned}$$

   (b) If the second ball is blue, what is the chance that the first ball has also been blue?

   Answer: $5/12$

   Solution: Using Bayes' rule,

$$\mathbb{P}(B* \mid *B) = \frac{\mathbb{P}(BB)}{\mathbb{P}(*B)} = \frac{(3/10)\cdot(5/12)}{3/10} = \frac{5}{12}\,.$$

(bonus) (c) Suppose we repeat the first step of the experiment over and over. In other words, we repeat drawing balls from the jar; each time, we record the color of the drawn ball and then put it back into the jar along with 2 more balls of the same color. What is the chance that the 8th ball we draw is blue?

   Answer: $3/10$

   Solution: Use induction to show that, for every n, the probability of drawing a blue ball at the n'th step is $3/10$.