Full Name: ______________________

Student No: ______________________

Grade:

For each answer, **provide a short argument**.

1. (3 points) Let X be a random variable with cumulative distribution function

$$F(x) := \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{if } x < -1, \\ 0.2 & \text{if } -1 \leq x < 1, \\ 0.7 & \text{if } 1 \leq x < 3, \\ 1 & \text{if } 3 \leq x. \end{cases}$$

(a) What is the probability that $0 \leq X \leq 1$?

Answer: 0.5

Solution: The only possible value for X within the interval $[0, 1]$ is 1. Therefore,

$$\mathbb{P}(0 \leq X \leq 1) = \mathbb{P}(X = 1) = \langle\text{amount of jump } F \text{ has at } 1\rangle = 0.7 - 0.2 = 0.5\ .$$

(b) Find the probability mass function of X.

Solution: The possible values of X are those at which F has a jump, that is, -1, 1 and 3. The probability that X takes each of its possible values is the amount of jump F has at that value. The jump at -1 is $0.2 - 0 = 0.2$, the jump at 1 is $0.7 - 0.2 = 0.5$ and the jump at 3 is $1 - 0.7 = 0.3$. Hence, the probability mass function of X is

$$p(x) = \mathbb{P}(X = x) = \begin{cases} 0.2 - 0 = 0.2 & \text{if } x = -1, \\ 0.7 - 0.2 = 0.5 & \text{if } x = 1, \\ 1 - 0.7 = 0.3 & \text{if } x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

As a sanity check, we can note that $0.2 + 0.5 + 0.3 = 1$ (the probabilities add up to 1) as it should be.

(c) Find the expected value of X.

Answer: 1.2

Solution: We can use the probability mass function obtained in the previous part to compute

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{P}(X = -1) \times (-1) + \mathbb{P}(X = 1) \times 1 + \mathbb{P}(X = 3) \times 3 \\ &= 0.2 \times (-1) + 0.5 \times 1 + 0.3 \times 3 = 1.2 \end{aligned}$$

2. (3 points) We repeat rolling a die until a 5 appears. Let N denote the number of rolls.

(a) What is the chance that $N > 5$?

Answer: $({}^5\!/_6)^5 \approx 0.4019$

Solution: The event $N > 5$ means "no 5 appears in the first 5 rolls". Therefore,

$$\mathbb{P}(N > 5) = \mathbb{P}(\text{no } 5 \text{ in the first } 5 \text{ rolls}) = ({}^5\!/_6)^5 .$$

(b) What is the expected value of N?

Answer: 6

Solution: Clearly, N is a geometric RV with parameter $p = 1/6$. Therefore, its expected value is $\mathbb{E}[N] = 1/p = 6$.

(c) What is the expected value of $1/3^N$?

Answer: ${}^1\!/_{13} \approx 0.07692$

Solution: Since $1/3^N$ is a function of N, we can find its expected value by summing over the possible values of N. Since N is geometric with parameter $1/6$, we have

$$\begin{aligned}\mathbb{E}[1/3^N] &= \sum_{k=1}^{\infty} \mathbb{P}(N = k) \cdot ({}^1\!/_{3^k}) \\ &= \sum_{k=1}^{\infty} ({}^5\!/_6)^{k-1} ({}^1\!/_6) \cdot ({}^1\!/_{3^k}) \\ &= \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{5}{6} \cdot \frac{1}{3}\right)^k .\end{aligned}$$

The latter is a geometric series with common factor $5/18$ and starting term $5/18$. Since $5/18$ is strictly between -1 and 1, the series converges and we get

$$\mathbb{E}[1/3^N] = \frac{1}{5} \times \frac{5/18}{1 - 5/18} = \frac{1}{13} .$$

3. (1 point) A bus carrying 10 passengers goes along a route with 7 designated stops. Each passenger gets off at a random stop, independently of the other passengers. Assuming that no new passenger enters the bus, what is the expected number of stops at which at least one passenger gets off of the bus?
[Hint: Introduce a Bernoulli RV for each designated stop.]

Answer: $7 \times \left[1 - ({}^6\!/_7)^{10}\right] \approx 5.5016$

Solution: Let N denote the number of stops at which at least one passenger gets off the bus. We are looking for $\mathbb{E}[N]$.

Number the stops $1, 2, 3, \ldots, 7$. For each stop k, let us define a Bernoulli RV X_k, where

$$X_k := \begin{cases} 1 & \text{if at least one passenger gets off at stop } k, \\ 0 & \text{otherwise.} \end{cases}$$

The parameter of this Bernoulli RV is

$$\mathbb{P}(X_k = 1) = 1 - \mathbb{P}(X_k = 0) = 1 - \mathbb{P}(\text{no passenger gets of at stop } k) = 1 - (6/7)^{10} .$$

Now, observe that $N = X_1 + X_2 + \cdots + X_7$. By the linearity of expectation, we get

$$\begin{aligned} \mathbb{E}[N] &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_7] \\ &= 7 \times \left[1 - (6/7)^{10}\right] \\ &\approx 5.5016 . \end{aligned}$$