Full Name: ______________________

Student No: ______________________

Instructor: Siamak Taati

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 8 questions, some with multiple parts, and a total score of 105 points. All answers require justifications.
- During the exam, you are allowed to use your phone to access the app for probability distributions (and possibly a calculator app), but the internet access of your phone must be switched off.
- The duration of the exam is 2 hours.

You can use the remainder of this page as scratch paper.

1. (15 points) Determine which of the following statements is True and which is False. In each case, give a short justification.

FALSE If A and B are two events with non-zero probabilities, then $\dfrac{\mathbb{P}(B \mid A)}{\mathbb{P}(A \mid B)} = \dfrac{\mathbb{P}(B)}{\mathbb{P}(A)}$.

> **Solution:** At first sight, we may notice that the claimed identity is just a rearrangement of the chain rule
> $$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B \mid A) = \mathbb{P}(B)\,\mathbb{P}(A \mid B)\ ,$$
> hence we may assume that it always holds. Namely, we obtain the claimed identity if we divide both sides of $\mathbb{P}(A)\,\mathbb{P}(B \mid A) = \mathbb{P}(B)\,\mathbb{P}(A \mid B)$ by $\mathbb{P}(A)\,\mathbb{P}(A \mid B)$. However, in order for this operation to be valid, $\mathbb{P}(A \mid B)$ must be non-zero.
> In fact, when $\mathbb{P}(A \mid B) = 0$, the left-hand side of the claimed identity becomes meaningless, so the identity does not hold. As an example, consider the random experiment of flipping a fair coin. Let A be the event that the coin comes up heads, and B the event that the coin comes up tails. Then, the right-hand side is $\dfrac{1/2}{1/2} = 1$ but the left-hand side is $\dfrac{0}{0}$, which is meaningless.

TRUE If $X \sim \mathsf{Bin}(n = 1024, p = 0.3)$ and $Y \sim \mathsf{Bin}(n = 999, p = 0.3)$ are two independent binomial random variables, then $X + Y$ is also a binomial random variable.

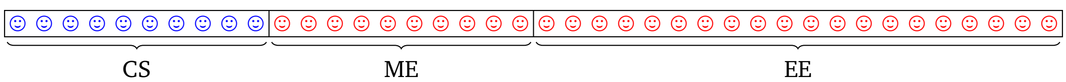
> **Solution:** Imagine X to be the number of successes in a sequence of 1024 independent trials each with probability 0.3 of success. Likewise, imagine Y to be the number of success in a sequence of 999 independent trials each with probability 0.3 of success. Imagine the two collections of trials to be independent. Then, $X+Y$ is the number of successes in $1024+999 = 2023$ independent trials each with probability 0.3 of success. Therefore, $X + Y \sim \mathsf{Bin}(n = 2023, p = 0.3)$.

FALSE Saying that 7 ± 0.5 is a 95% confidence interval for a parameter θ means $\mathbb{P}(6.5 < \theta < 7.5) = 95\%$.

> **Solution:** Note that the parameter θ is a fixed non-random value, albeit unknown to us. Hence, $\mathbb{P}(6.5 < \theta < 7.5)$ is either 1 or 0, depending on whether θ is in the interval 7 ± 0.5 or not.
> The correct interpretation is that 7 ± 0.5 is generated by plugging in the observed values of a sample in an interval estimator (L, R) for which we have $\mathbb{P}(L < \theta < R) = 0.95$.

2. (10 points) A statistics class contains 40 students, of which 10 are computer science majors, 10 are mechanical engineering majors, and 20 are electrical engineering majors. A group of 5 distinct students are chosen from this class at random.

   (a) What is the probability that all the group members have the same major?

> **Solution:**
>
> 
>
>
> The sample space Ω is the collection of all 5-element subsets of the set of all 40 students in class. Since the 5 students are chosen at random, all the outcomes in Ω are equally likely, each having a probability of $1/|\Omega|$. The event E that we are interested in is identified by those

5-element sets that are entirely included in one of the three categories of computer science, mechanical engineering, and electrical engineering majors. We have

$$|\Omega| = \binom{40}{5}, \qquad |E| = \binom{10}{5} + \binom{10}{5} + \binom{20}{5},$$

hence

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \boxed{\frac{\binom{10}{5} + \binom{10}{5} + \binom{20}{5}}{\binom{40}{5}}} = \frac{252 + 252 + 15504}{658008} \approx 0.02433 .$$

(b) What is the expected number of computer science majors in the group?
[*Hint*: Use the linearity of expectation.]

Solution: Let N be the (random) number of computer science majors that end up in the chosen group. We wish to compute $\mathbb{E}[N]$.
Observe that N can be written as

$$N = X_1 + X_2 + \cdots + X_{10}$$

where

$$X_k = \begin{cases} 1 & \text{if the } k\text{th CS major is in the chosen group,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for each k,

$$\mathbb{E}[X_k] = \mathbb{P}(X_k = 1) = \mathbb{P}(\text{the } k\text{th CS major is in the chosen group}) = \frac{\binom{39}{4}}{\binom{40}{5}} = \frac{5}{40} .$$

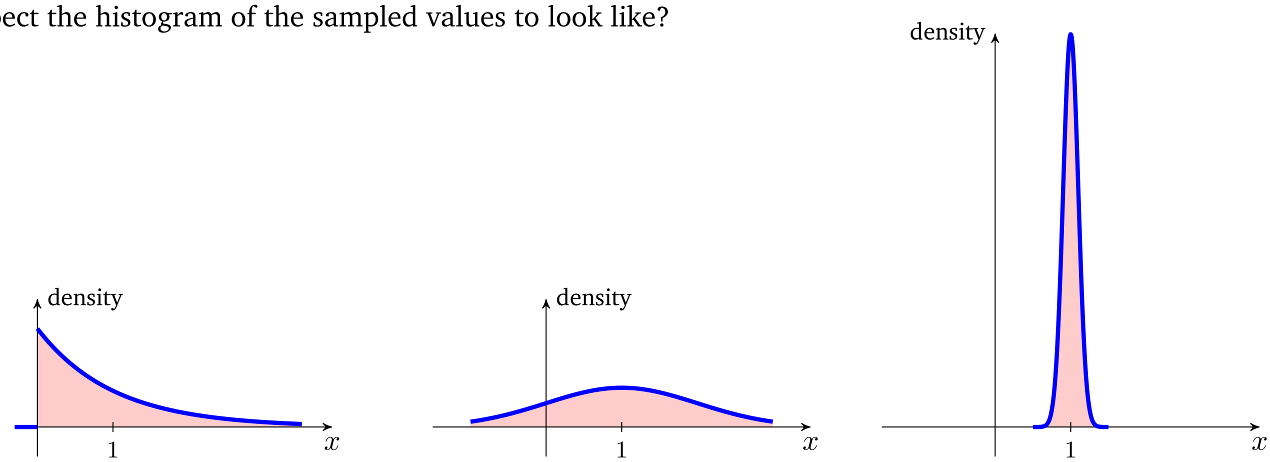
It follows from the linearity of expectation that

$$\begin{aligned} \mathbb{E}[N] &= \mathbb{E}[X_1 + X_2 + \cdots + X_{10}] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{10}] \\ &= 10 \times \frac{5}{40} \\ &= \boxed{5/4} = 1.25 . \end{aligned}$$

Alternatively, we may notice that N has a hypergeometric distribution with parameters $N = 40$ (total number of balls), $M = 10$ (total number of blue balls), and $n = 5$ (number of drawn balls), and remember

$$\mathbb{E}[N] = n\frac{M}{N} = 5\frac{10}{40} = 5/4 .$$

3. (5 points) We draw a sample of size 100 from an exponential distribution with rate 1. How do you expect the histogram of the sampled values to look like?



(a) (b) (c)

Solution: The histogram would resemble the sketch in option (a). Namely, the histogram would approximate the probability density function of the exponential distribution with rate 1, which is exactly what is depicted in sketch (a).

4. (15 points) Let $\Theta_1, \Theta_2, \ldots, \Theta_{100}$ be independent and identically distributed random variables, each having uniform distribution over the interval $[0, \pi]$.

   (a) What is the expected value of $\sin(\Theta_1)$?

Solution: This question is identical to Problem 1(b) on the second midterm. Since Θ_1 is uniformly distributed over $[0, \pi]$, its pdf is

$$f_\Theta(\theta) = \begin{cases} 1/\pi & \text{if } 0 \le \theta \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\mathbb{E}[\sin(\Theta)] = \int_{-\infty}^{\infty} \sin(\theta) f_\Theta(\theta)\,\mathrm{d}\theta = \int_0^\pi \sin(\theta)\frac{1}{\pi}\,\mathrm{d}\theta = \frac{-\cos(\theta)}{\pi}\bigg|_{\theta=0}^{\pi} = \boxed{2/\pi} \approx 0.6366\,.$$

   (b) What is the variance of $\sin(\Theta_1)$? [*Recall*: $(\sin\theta)^2 = \dfrac{1-\cos(2\theta)}{2}$.]

Solution: Again, using the fact that Θ_1 is uniform, we have

$$\begin{aligned}
\mathbb{E}[\sin(\Theta_1)^2] &= \int_{-\infty}^{\infty} \sin(\theta)^2 f_\Theta(\theta)\,\mathrm{d}\theta \\
&= \int_0^\pi \sin(\theta)^2 \frac{1}{\pi}\,\mathrm{d}\theta \\
&= \int_0^\pi \frac{1-\cos(2\theta)}{2}\frac{1}{\pi}\,\mathrm{d}\theta && \text{(using the recalled trigonometric identity)} \\
&= \frac{1}{2\pi}\theta\bigg|_{\theta=0}^{\pi} - \frac{1}{4\pi}\sin(2\theta)\bigg|_{\theta=0}^{\pi} \\
&= 1/2\,.
\end{aligned}$$

Using this and the result of the previous part, we get

$$\mathbb{V}\mathrm{ar}[\sin(\Theta_1)] = \mathbb{E}[\sin(\Theta_1)^2] - \mathbb{E}[\sin(\Theta)]^2 = \boxed{\frac{1}{2} - \frac{4}{\pi^2}} \approx 0.09471527 \,.$$

Let $\overline{Y} = \frac{1}{100}\sum_{k=1}^{100} \sin(\Theta_k)$.

(c) What is the approximate probability that $\overline{Y} > 0.65$?

Solution: For $k = 1, 2, \ldots, 100$, let $Y_k = \sin(\Theta_k)$. Since $\Theta_1, \Theta_2, \ldots, \Theta_{100}$ are i.i.d., so are $Y_1, Y_2, \ldots, Y_{100}$. By the Central Limit Theorem, the average $\overline{Y} = \frac{1}{100}\sum_{k=1}^{100} Y_k$ is approximately normally distributed with parameters

$$\begin{aligned}
\mathbb{E}[\overline{Y}] &= \mathbb{E}\left[\frac{1}{100}\sum_{k=1}^{100} Y_k\right] \\
&= \frac{1}{100}\sum_{k=1}^{100} \mathbb{E}[Y_k] && \text{(linearity of expectation)} \\
&= 2/\pi && \text{(by the result of part (a))} \\
&\approx 0.6366 \,.
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[\overline{Y}] &= \mathbb{V}\mathrm{ar}\left[\frac{1}{100}\sum_{k=1}^{100} Y_k\right] \\
&= \frac{1}{100^2}\,\mathbb{V}\mathrm{ar}\left[\sum_{k=1}^{100} Y_k\right] \\
&= \frac{1}{100^2}\sum_{k=1}^{100} \mathbb{V}\mathrm{ar}[Y_k] && \text{(independence of } Y_1, Y_2, \ldots, Y_{100}\text{)} \\
&= \frac{1}{100}\left(\frac{1}{2} - \frac{4}{\pi^2}\right) && \text{(by the result of part (b))}
\end{aligned}$$

$$\begin{aligned}
\mathbb{SD}[\overline{Y}] &= \sqrt{\frac{1}{100}\left(\frac{1}{2} - \frac{4}{\pi^2}\right)} \\
&\approx 0.03078 \,.
\end{aligned}$$

Therefore, using the app for normal distribution with parameters $\mu = 0.6366$ and $\sigma = 0.3078$, we can compute

$$\mathbb{P}(\overline{Y} > 0.65) \approx \boxed{0.33166} \,.$$

5. (15 points) Two random variables X and Y have joint probability density

$$f(x,y) = \begin{cases} \dfrac{1}{(x+y)^3} & \text{if } x, y > c, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a positive constant.

(a) Find c.

Solution: Since f is a joint probability density function, its integral over the entire plane must be 1. We have

$$\begin{aligned} 1 &= \iint_{\mathbb{R}\times\mathbb{R}} f(x,y)\,\mathrm{d}A \\ &= \int_{y=c}^{\infty}\int_{x=c}^{\infty} (x+y)^{-3}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=c}^{\infty} \left[\frac{-1}{2}(x+y)^{-2}\right]_{x=c}^{\infty} \mathrm{d}y \\ &= \int_{y=c}^{\infty} \frac{1}{2}(c+y)^{-2}\,\mathrm{d}y \\ &= \left[\frac{-1}{2}(c+y)^{-1}\right]_{y=c}^{\infty} \\ &= \frac{1}{4c}\,. \end{aligned}$$

Therefore, $\boxed{c = 1/4}$.

(b) Find the marginal probability densities of X and Y.

Solution: Note that X can only take values larger than $c = 1/4$. For $x > 1/4$, the pdf of X is given by

$$\begin{aligned} f_X(x) &= \int_{y=-\infty}^{\infty} f(x,y)\,\mathrm{d}y \\ &= \int_{y=c}^{\infty} (x+y)^{-3}\,\mathrm{d}y \\ &= \left[-\frac{1}{2}(x+y)^{-2}\right]_{y=c}^{\infty} \\ &= \frac{1}{2(x+c)^2} \\ &= \frac{8}{(4x+1)^2}\,. \end{aligned}$$

In summary,

$$f_X(x) = \boxed{\begin{cases} \dfrac{8}{(4x+1)^2} & \text{if } x > 1/4, \\ 0 & \text{if } x < 1/4. \end{cases}}$$

By symmetry,

$$f_Y(y) = \boxed{\begin{cases} \dfrac{8}{(4y+1)^2} & \text{if } y > 1/4, \\ 0 & \text{if } y < 1/4. \end{cases}}$$

(c) Find the probability density of $\min(X, Y)$.

Solution: There are two slightly different approaches.

Approach 1: (long but instructive)
Let $Z := \min(X, Y)$. As we did for similar examples in class, we start by deriving an expression for the cdf of Z.
Denoting the joint cdf of X and Y by F and their marginal cdfs by F_X and F_Y, we can write the cdf of Z as

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) \\
&= \mathbb{P}(\min(X,Y) \leq z) \\
&= \mathbb{P}(X \leq z \text{ or } Y \leq z) && (\min(X,Y) \leq z \text{ if and only if} \\
& && \quad \text{either } X \leq z \text{ or } Y \leq z) \\
&= \mathbb{P}(X \leq z) + \mathbb{P}(Y \leq z) && \text{(inclusion-exclusion principle)} \\
&\qquad - \mathbb{P}(X \leq z \text{ and } Y \leq z) \\
&= F_X(z) + F_Y(z) - F(z,z)
\end{aligned}$$

Differentiating both sides, we obtain

$$f_Z(z) = f_X(z) + f_Y(z) - \frac{\mathrm{d}}{\mathrm{d}z} F(z,z) \; . \tag{⚀}$$

From the previous parts, we know the values of $f_X(z)$ and $f_Y(z)$. To find the value of the last term, we can use the chain rule to write

$$\frac{\mathrm{d}}{\mathrm{d}z} F(z,z) = F_1(z,z) \cdot \frac{\mathrm{d}}{\mathrm{d}z} z + F_2(z,z) \cdot \frac{\mathrm{d}}{\mathrm{d}z} z \; , \tag{⚁}$$

where

$$F_1(x,y) = \frac{\partial}{\partial x} F(x,y) \; , \qquad\qquad F_2(x,y) = \frac{\partial}{\partial y} F(x,y) \; .$$

When $x < c$ or $y < c$, we have $F_1(x,y) = F_2(x,y) = 0$ (why?). For $x, y > c$, we have

$$\begin{aligned}
F_1(x,y) &= \frac{\partial}{\partial x} \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u,v) \,\mathrm{d}v \,\mathrm{d}u \\
&= \int_{v=-\infty}^{y} f(x,v) \,\mathrm{d}v && \text{(fundamental theorem of calculus)} \\
&= \int_{v=c}^{y} \frac{1}{(x+v)^3} \,\mathrm{d}v \\
&= \left[-\frac{1}{2(x+v)^2} \right]_{v=c}^{y} \\
&= \frac{1}{2(x+c)^2} - \frac{1}{2(x+y)^2} \; ,
\end{aligned}$$

hence

$$F_1(z,z) = \frac{1}{2(z+c)^2} - \frac{1}{8z^2} \; .$$

By symmetry,

$$F_2(z,z) = \frac{1}{2(z+c)^2} - \frac{1}{8z^2} \; .$$

Substituting in (⚁), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} F(z,z) = \begin{cases} \dfrac{1}{(z+c)^2} - \dfrac{1}{4z^2} & \text{if } z > c, \\ 0 & \text{if } z < c. \end{cases} \tag{⚂}$$

For $z < c$, we have $f_Z(z) = 0$, while for $z > c$, the expressions (⚀) and (⚂) give

$$f_Z(z) = \frac{1}{2(z+c)^2} + \frac{1}{2(z+c)^2} - \frac{1}{(z+c)^2} + \frac{1}{4z^2} = \frac{1}{4z^2} \;.$$

In summary,

$$f_Z(z) = \boxed{\begin{cases} \dfrac{1}{4z^2} & \text{if } z > 1/4, \\ 0 & \text{if } z < 1/4. \end{cases}}$$

<u>Approach 2</u> (short and efficient):
Let $Z := \min(X, Y)$. Let us start with computing the cdf of Z.
Denoting the joint cdf of X and Y by F and their marginal cdfs by F_X and F_Y, we can write the cdf of Z as

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) \\ &= \mathbb{P}(\min(X,Y) \leq z) \\ &= 1 - \mathbb{P}(\min(X,Y) > z) \\ &= 1 - \mathbb{P}(X > z \text{ and } Y > z) && (\min(X,Y) > z \text{ if and only if both } X > z \text{ or } Y > z) \\ &= 1 - \iint_{\substack{x>z \\ y>z}} f(x,y)\,\mathrm{d}A \end{aligned}$$

Since X and Y can only take values larger than c, the same is true for Z. Hence, $F_Z(z) = 0$ for $z \leq c = 1/4$. For $z > c$, the same computation as in part (a) (substituting z for c) gives

$$\iint_{\substack{x>z \\ y>z}} f(x,y)\,\mathrm{d}A = \frac{1}{4z} \;.$$

Therefore,

$$F_Z(z) = \begin{cases} 1 - \dfrac{1}{4z} & \text{if } z > 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

Differentiation gives

$$f_Z(z) = \boxed{\begin{cases} \dfrac{1}{4z^2} & \text{if } z > 1/4, \\ 0 & \text{if } z < 1/4. \end{cases}}$$

6. (10 points) You are the first passenger of a minibus at the Cola intersection. The driver promises to leave as soon as either he has acquired 4 more passengers or a period of 15 minutes has passed since your arrival. Passengers arrive according to a Poisson process with an average rate of one passenger every three minutes.

   (a) What is the probability that you have to wait 15 minutes before the minibus leaves?

Solution: This problem is adapted form Example 4.5 of the book *Understanding Probability* by Henk Tijms.

Let N denote the number of passengers arriving within 15 minutes after your arrival. Since the passengers arrive according to a Poisson process with a rate of $\lambda = 1/3$ per minutes, N is a Poisson random variable with a mean of $\mu = 15 \times 1/3 = 5$. Therefore,

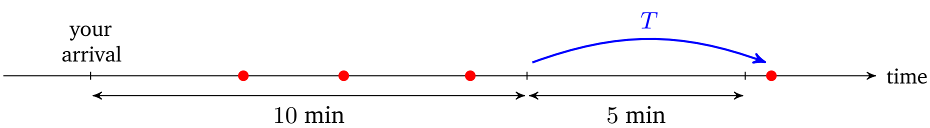
$$\begin{aligned}
&\mathbb{P}(\text{you have to wait } 15 \text{ minutes})\\
&\quad = \mathbb{P}(N < 4)\\
&\quad = \mathbb{P}(N = 0) + \mathbb{P}(N = 1) + \mathbb{P}(N = 2) + \mathbb{P}(N = 3)\\
&\quad = \mathrm{e}^{-5}\frac{5^0}{0!} + \mathrm{e}^{-5}\frac{5^1}{1!} + \mathrm{e}^{-5}\frac{5^2}{2!} + \mathrm{e}^{-5}\frac{5^3}{3!}\\
&\quad \approx \boxed{0.26503}\,.
\end{aligned}$$

Alternatively, the arrival time S_4 of the fourth passenger has a gamma distribution with shape $\alpha = 4$ and rate $\lambda = 1/3$. Hence, we can use the app for gamma distribution to compute

$$\mathbb{P}(\text{you have to wait } 15 \text{ minutes} = \mathbb{P}(S_4 \geq 15\,\text{min}) \approx \boxed{0.26503}\,.$$

   (b) Ten minutes pass since your arrival, and in the meantime, three other passengers arrive. What is the probability that you will have to wait five more minutes before the minibus leaves?

Solution:



Let T denote the waiting time until the next passenger arrives starting from 10 minutes after your arrival. Then, T has an exponential distribution with rate $\lambda = 1/3$, irrespective of the number of arrivals in the first ten-minute interval. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\begin{array}{c|c}\text{you have to wait} & \text{3 passengers have arrived}\\ \text{5 more minutes} & \text{within the first 10 minutes}\end{array}\right)\\
&\quad = \mathbb{P}(T > 5\,\text{min})\\
&\quad = \mathrm{e}^{-\lambda \times 5}\\
&\quad = \mathrm{e}^{-5/3}\\
&\quad \approx \boxed{0.18888}\,.
\end{aligned}$$

Alternatively, we can proceed as follows. Let N_1 denote the number of passengers arriving within the first 10 minutes, and N_2 the number of passengers arriving within the next 5 minutes. Then, N_1 and N_2 are independent Poisson random variables with means $\mu_1 = 10\lambda = 10/3$ and $\mu_2 = 5\lambda = 5/3$, respectively. Therefore,

$$\mathbb{P}\left(\begin{array}{c|c}\text{you have to wait} & \text{3 passengers have arrived}\\ \text{5 more minutes} & \text{within the first 10 minutes}\end{array}\right)$$

$$
\begin{aligned}
&= \mathbb{P}(N_2 = 0 \mid N_1 = 3) \\
&= \mathbb{P}(N_2 = 0) && \text{(by independence)} \\
&= \mathrm{e}^{-\mu_2} \frac{\mu_2^0}{0!} \\
&= \mathrm{e}^{-5/3} \\
&\approx \boxed{0.18888}\,.
\end{aligned}
$$

7. (15 points) We have a (possibly biased) coin with parameter p. Let Y denote the number of heads in n flips of the coin, and consider the point estimator $\widehat{p}_n := Y/n$ for p.

   (a) Argue that $\widehat{p}_n$ is consistent.

Solution: We can write $Y = X_1 + X_2 + \cdots + X_n$, where

$$
X_k = \begin{cases} 1 & \text{if in the } k\text{th flip, the coin comes up heads,} \\ 0 & \text{otherwise,} \end{cases}
$$

for each $k = 1, 2, \ldots, n$. The random variables $X_1, X_2, \ldots, X_n$ are i.i.d., each having a Bernoulli distribution with parameter p. Hence, according to the law of large numbers,

$$
\widehat{p}_n = \frac{1}{n} Y = \frac{1}{n} \sum_{k=1}^{n} X_k
$$

converges to $\mathbb{E}[X_1] = p \times 1 + (1-p) \times 0 = p$. Therefore, $\widehat{p}_n$ is consistent.

   (b) Argue that $\widehat{p}_n$ is unbiased.

Solution: By the linearity of expectation, we have

$$
\mathbb{E}[\widehat{p}_n] = \mathbb{E}\left[\frac{1}{n} \sum_{k=1}^{n} X_k\right] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[X_k] = \frac{1}{n} \sum_{k=1}^{n} p = p\,.
$$

Hence, $\widehat{p}_n$ is unbiased.

   (c) Compute the mean squared error $\mathsf{MSE}(\widehat{p}_n) = \mathbb{E}[(\widehat{p}_n - p)^2]$.

Solution: Since $\widehat{p}_n$ is unbiased,

$$
\begin{aligned}
\mathsf{MSE}[\widehat{p}_n] &= \mathbb{V}\mathrm{ar}[\widehat{p}_n] \\
&= \mathbb{V}\mathrm{ar}\left[\frac{1}{n} \sum_{k=1}^{n} X_k\right] \\
&= \frac{1}{n^2} \mathbb{V}\mathrm{ar}\left[\sum_{k=1}^{n} X_k\right] \\
&= \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}[X_k] && \text{(by independence)} \\
&= \frac{1}{n} \mathbb{V}\mathrm{ar}[X_1]\,.
\end{aligned}
$$

For the Bernoulli random variable X_1, we have

$$\mathbb{V}\mathrm{ar}[X_1] = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = p - p^2 = p(1-p) \; .$$

Hence,

$$\mathsf{MSE}[\widehat{p}_n] = \boxed{\frac{p(1-p)}{n}} \, .$$

8. (20 points) The weight of a randomly picked pencil from a certain brand can be assumed to be normally distributed with mean μ and variance σ^2. We would like to estimate μ. Let W_1, W_2, W_3, W_4 denote the weights of a random sample of four pencils and

$$\overline{W} = \frac{W_1 + W_2 + W_3 + W_4}{4} \qquad \text{and} \qquad S^2 = \frac{1}{3}\sum_{i=1}^{4}(W_i - \overline{W})^2$$

their sample mean and sample variance.

(a) Identify $\mathbb{E}[\overline{W}]$ and $\mathbb{V}\mathrm{ar}[\overline{W}]$.

Solution: We have

$$\begin{aligned}
\mathbb{E}[\overline{W}] &= \mathbb{E}\left[\frac{W_1 + W_2 + W_3 + W_4}{4}\right] \\
&= \frac{1}{4}\big(\mathbb{E}[W_1] + \mathbb{E}[W_2] + \mathbb{E}[W_3] + \mathbb{E}[W_4]\big) \qquad \text{(linearity of expectation)} \\
&= \frac{1}{4}(\mu + \mu + \mu + \mu) \\
&= \boxed{\mu} \, ,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[\overline{W}] &= \mathbb{V}\mathrm{ar}\left[\frac{W_1 + W_2 + W_3 + W_4}{4}\right] \\
&= \frac{1}{16}\,\mathbb{V}\mathrm{ar}\left[W_1 + W_2 + W_3 + W_4\right] \\
&= \frac{1}{16}\big(\mathbb{V}\mathrm{ar}[W_1] + \mathbb{V}\mathrm{ar}[W_2] + \mathbb{V}\mathrm{ar}[W_3] + \mathbb{V}\mathrm{ar}[W_4]\big) \qquad \text{(independence)} \\
&= \frac{1}{16}(\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2) \\
&= \boxed{\frac{1}{4}\sigma^2} \, .
\end{aligned}$$

(b) What is the distribution of $\overline{W}$?

Solution: The random variable $\overline{W}$ is a linear combination of four independent normal random variables. Therefore, by the stability of the normal distribution, $\overline{W}$ is also normally distributed. Its parameters are the ones obtained in part (a).

(c) Based on this sample, construct an interval estimator for μ that has a confidence level of 94%.

Solution: Let

$$Z = \frac{\overline{W} - \mu}{\sigma/2}\,, \qquad \text{and} \qquad T = \frac{\overline{W} - \mu}{S/2}\,.$$

We know that Z has the standard normal distribution, whereas T has the t-distribution with $4 - 1 = 3$ degrees of freedom. Since we do not have access to σ, we use T instead of Z.
For every $a > 0$, we have

$$\begin{aligned}
\mathbb{P}\left(\overline{W} \pm aS/2 \text{ contains } \mu\right) &= \mathbb{P}\left(-aS/2 < \overline{W} - \mu < aS/2\right) \\
&= \mathbb{P}\left(-a < \frac{\overline{W} - \mu}{S/2} < a\right) \\
&= \mathbb{P}(-a < T < a) \\
&= F_{\mathsf{T}(3)}(a) - F_{\mathsf{T}(3)}(-a) \\
&= 2F_{\mathsf{T}(3)}(a) - 1\,,
\end{aligned}$$

where $F_{\mathsf{T}(3)}$ denotes the cdf of the t-distribution with 3 degrees of freedom. Hence, for every $a > 0$, we obtain an interval estimator $\overline{W} \pm aS/2$ for μ that has confidence level $2F_{\mathsf{T}(3)}(a) - 1$. In order to have a confidence level of 94%, we must have $2F_{\mathsf{T}(3)}(a) - 1 = 94\%$, or equivalently $F_{\mathsf{T}(3)}(a) = 0.97$. Using the app for the t-distribution with 3 degrees of freedom, we find $a \approx 2.95051$. Thus, $aS/2 \approx 1.475255S$.
In conclusion, we find the interval estimator

$$\boxed{\overline{W} \pm 1.475255S \qquad \text{(with 94\% confidence)}}$$

for the mean weight μ of a pencil.

Suppose we measure the weight of four pencils and obtain the values

$$9.96\,\mathrm{g} \qquad 9.55\,\mathrm{g} \qquad 9.73\,\mathrm{g} \qquad 9.87\,\mathrm{g}$$

which have a mean of $9.7775\,\mathrm{g}$ and a standard deviation of $0.17877\,\mathrm{g}$.

(d) Use the interval estimator obtained in part (c) to give a 94% confidence interval for the mean weight of a pencil.

Solution: Substituting the observed values $\overline{W} = 9.7775\,\mathrm{g}$ and $S = 0.17877\,\mathrm{g}$ in the above interval estimator, we obtain the estimate

$$\boxed{9.7775\,\mathrm{g} \pm 0.2637\,\mathrm{g} \qquad \text{(with 94\% confidence)}}$$

for the mean weight μ of a pencil.