Full Name: ______________________

Student No: ______________________

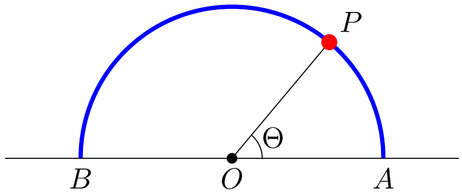
Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 6 questions, some with multiple parts, and a total score of 70 points. All answers require justifications.
- The duration of the exam is 1h30.

You can use the remainder of this page as scratch paper.

1. (15 points) A point P is picked uniformly at random from a semicircle. Let Θ denote the angle of this point relative to one of the endpoints of the semicircle.



(a) What is the probability that $\sin(\Theta) > \frac{1}{2}$?

(b) What is the expected value of $\sin(\Theta)$?

(c) What is the expected value of $\cos(\Theta)$?

2. (20 points) A random variable X has the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{if } x < -5, \\ 0.3 & \text{if } -5 \le x < -1, \\ 0.55 & \text{if } -1 \le x < 1, \\ 0.8 & \text{if } 1 \le x < 3, \\ 1 & \text{if } 3 \le x. \end{cases}$$

(a) What is the probability that $X > 0$?

(b) What is the probability that $1 \le X \le 3$?

(c) Find the expected value of X.

(d) Find the variance of X.

3. (15 points) A continuous random variable W has the following probability density function:

$$f(w) = \begin{cases} 0 & \text{if } w < -3, \\ 0.1 & \text{if } -3 < w < 0, \\ a\,\mathrm{e}^{-2w} & \text{if } 0 < w, \end{cases}$$

where a is an unknown constant.

(a) Find a.

(b) What is the probability that $W \leq 1$?

(c) What is the expected value of W?

4. (5 points) Let X be a normal random variable with mean 3 and variance 4. Express the probability that $2 \leq X \leq 5$ either in terms of the cumulative distribution function Φ of the standard normal distribution, or as an integral.

5. (5 points) Let G be a discrete random variable with the following probability mass function:

$$p(n) = \begin{cases} (5/6) \times (2/3)^n & \text{if } n = 1, 3, 5, 7, \ldots \text{ (a positive odd integer)}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of G.

6. (10 points) A box contains 10 cards numbered 1 to 10. You are offered to play the following game: At the beginning of the game, you pay an entrance fee of $\$3$. You then draw 6 cards from the box, one after another, with replacement. You receive $\$1$ for each distinct number you draw. For instance, if you draw $1, 6, 4, 4, 3, 4$, you receive $\$4$ for the distinct numbers $1, 3, 4, 6$.

   (a) Calculate your expected payoff. [*Hint*: Use the linearity of expectation.]

   (b) Is this a fair game? If not, how would you change the entrance fee in order to make it fair?

You can use this sheet as extra space for your solutions.

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