Full Name: ______________________

Student No: ______________________

Grade:

For each answer, **provide a short argument**.

1. (2 points) A bag contains 1000 coins. One of the coins has heads drawn on both sides, but all the other coins are fair coins. We pick a coin from the bag at random and flip it 8 times.

   (a) What is the probability that in all 8 flips, the coin comes up heads?

   Answer: ≈ 0.004902

   Solution: Following the usual practice, let us start by choosing the sample space. Each possible outcome can be adequately described by two pieces of information: (1) whether the picked coin is the two-headed one or not, and (2) the result of the 8 flips of that coin. Thus, a suitable sample space for this experiment is

   $$\begin{aligned}\Omega &= \big\{(\text{two-headed}, \texttt{HHHHHHHH}), (\text{fair}, \texttt{HHHHHHHH}), (\text{fair}, \texttt{HHHHHHHT}), \ldots (\text{fair}, \texttt{TTTTTTTT})\big\} \\ &= \big\{(\text{two-headed}, \texttt{H}^8)\big\} \cup \big(\{\text{fair}\} \times \{\texttt{H}, \texttt{T}\}^8\big)\end{aligned}$$

   Let E denote the event that, in all 8 flips, the picked coin comes up heads, and let C denote the event that the picked coin is two-headed. Then,

   $$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(C \cap E) + \mathbb{P}(C^c \cap E) && \text{(principle of total probability)} \\ &= \mathbb{P}(C)\,\mathbb{P}(E \mid C) + \mathbb{P}(C^c)\,\mathbb{P}(E \mid C^c) && \text{(chain rule)} \\ &= \frac{1}{1000} \times 1 + \frac{999}{1000} \times \left(\frac{1}{2}\right)^8 \\ &\approx 0.004902\ .\end{aligned}$$

   An alternative (but equivalent) approach is to draw the tree of possibilities. The first branching corresponds to the choice of the coin, and the second corresponds to whether the coin comes up heads in all 8 flips or not.

   (b) If the 8 flips all resulted in heads, what is the probability that we have picked the two-headed coin?

   Answer: ≈ 0.2040

   Solution: We are looking for $\mathbb{P}(C \mid E)$. Using the result of the previous part, we can write

   $$\mathbb{P}(C \mid E) = \frac{\mathbb{P}(C \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(C)\,\mathbb{P}(E \mid C)}{\mathbb{P}(E)} = \frac{\frac{1}{1000} \times 1}{\frac{1}{1000} \times 1 + \frac{999}{1000} \times \left(\frac{1}{2}\right)^8} \approx 0.2040\ .$$

2. (2 points) You and your friend are in a class with 30 students. Today's class involves two activities. For the first activity, the professor randomly splits the class into two groups A and B of 15 students each.

(a) What is the probability that you and your friend are in the same group?

Answer: $14/29$

Solution: The sample space Ω_1 is the set of all ways one can split a class of 30 students into two groups of equal sizes. There are $\binom{30}{15}$ possibilities (why?), all being equally likely.
Let E denote the event that you and your friend are in the same group. For this to happen, you should either be both in group A or both in group B. The number of possibilities in which you and your friend are in group A is $\binom{28}{13}$ (why?), hence in total, there are $2\binom{28}{13}$ possible outcomes that realize E.
It follows that

$$\mathbb{P}(E) = \frac{|E|}{|\Omega_1|} = \frac{2\binom{28}{13}}{\binom{30}{15}} = \frac{2 \times \frac{28!}{13!15!}}{\frac{30!}{15!15!}} = \frac{2 \times 15 \times 14}{30 \times 29} = \frac{14}{29} \,.$$

An alternative approach is through the principle of total probability. Namely, you will either be in group A or in group B. Conditioned on each of these two possibilities, the probability that your friend is in the same group is $14/29$. Hence, in total, the probability that your friend is in the same group as you is $14/29$.

For the second activity, the professor randomly pairs the students in group A with the students in group B.

(b) What is the probability that you and your friend are paired?

Answer: $1/29$

Solution: Since the experiment is extended, it requires a new sample space. Each possible outcome is identified by two pieces of information: (1) the splitting of the class into groups A and B, and (2) the pairing of students in A with the students in B.
Let F denote the event that you and your friend are paired, and E as in the previous part. Using the principle of total probability and the chain rule, we have

$$\mathbb{P}(F) = \mathbb{P}(E)\,\mathbb{P}(F \mid E) + \mathbb{P}(E^{\mathsf{c}})\,\mathbb{P}(F \mid E^{\mathsf{c}}) \,.$$

If you and your friend are in the same group, then clearly you cannot be paired, hence $\mathbb{P}(F \mid E) = 0$. If you and your friend are in different groups, then out of $15!$ equally likely different pairing of the members of A and B, there are exactly $14!$ in which you and your friend are paired. Thus, $\mathbb{P}(F \mid E^{\mathsf{c}}) = 14!/15! = 1/15$. It follows that

$$\mathbb{P}(F) = \mathbb{P}(E)\,\underbrace{\mathbb{P}(F \mid E)}_{0} + \underbrace{\mathbb{P}(E^{\mathsf{c}})}_{1-\mathbb{P}(E)}\,\underbrace{\mathbb{P}(F \mid E^{\mathsf{c}})}_{1/15} = 0 + \frac{15}{29} \times \frac{1}{15} = \frac{1}{29} \,.$$

An alternative way to find $\mathbb{P}(F)$ is to note that in order to pair the students the professor does not need to first split them into two groups. Instead, she/he can use the list of the students to pair them in a step-by-step manner as follows. At each step, the professor picks the first unpaired student on the list and pairs him/her with another student chosen at random among the unpaired ones. This procedure would be equivalent to the previous method insofar as the pairing is concerned. Namely, all possible pairings of the 30 students in the class are equally likely to be achieved.

Now, note that the order of the list of the students is irrelevant, so we can, without loss of generality, use a list in which you are the first student. Then it becomes apparent that the probability that your friend is paired with you is $1/29$.

3. (0 points) On the scale of 0–4, what is your estimate of your grade on this quiz?