Full Name: ______________________

Student No: ______________________

Grade:

For each answer, **provide a short argument**.

1. (3 points) Let X be a random variable with cumulative distribution function

$$F(x) := \mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x < -2, \\ 0.25 & \text{if } -2 \le x < -1, \\ 0.6 & \text{if } -1 \le x < 1, \\ 0.9 & \text{if } 1 \le x < 2, \\ 1 & \text{if } 2 \le x. \end{cases}$$

(a) What is the probability that $-1.5 < X \le 1.5$?

Answer: 0.65

Solution: We have

$$\begin{aligned} \mathbb{P}(-1.5 < X \le 1.5) &= \mathbb{P}(X \le 1.5) - \mathbb{P}(X \le -1.5) \\ &= F(1.5) - F(-1.5) = 0.9 - 0.25 = 0.65\ . \end{aligned}$$

(b) What is the probability that $X \ge 1$?

Answer: 0.4

Solution: From its cdf, it can be seen that X is a discrete random variable (why?) with possible values -2, -1, 1 and 2. We have

$$\mathbb{P}(X \ge 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X \le -1) = 1 - F(-1) = 1 - 0.6 = 0.4\ .$$

(c) Find the expected value of X.

Answer: -0.35

Solution: The probability that X takes each of its possible values is the amount of jump F has at that value. In particular, $\mathbb{P}(X = -2) = 0.25$, $\mathbb{P}(X = -1) = 0.6 - 0.25 = 0.35$, $\mathbb{P}(X = 1) = 0.9 - 0.6 = 0.3$, and $\mathbb{P}(X = 2) = 1 - 0.9 = 0.1$. Therefore,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{P}(X = -2) \cdot (-2) + \mathbb{P}(X = -1) \cdot (-1) + \mathbb{P}(X = 1) \cdot 1 + \mathbb{P}(X = 2) \cdot 2 \\ &= 0.25 \times (-2) + 0.35 \times (-1) + 0.3 \times 1 + 0.1 \times 2 = -0.35\ . \end{aligned}$$

2. (2 points) We distribute 22 books into 9 boxes at random.

   (a) What is the probability that box #1 contains more than 3 books?

   Answer:
   ≈ 0.22321

   Solution: Let X denote the number of books in box #1. Each book can either be placed in box #1 (in which case, it will be counted in X) or not (in which case, it will not be counted in X). Thus, we can think of X as the number of successful trials among 22 trials, one for each book, where "success" means being placed in box #1. These trials are independent, each with a success probability of $1/9$. Thus, X has a binomial distribution with parameters $n = 22$ and $p = 1/9$. In particular,

   $$\begin{aligned}\mathbb{P}(X > 3) &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) - \mathbb{P}(X = 3) \\ &= 1 - (8/9)^{22} - 22(1/9)(8/9)^{21} - \binom{22}{2}(1/9)^2(8/9)^{20} - \binom{22}{3}(1/9)^3(8/9)^{19} \quad \approx 0.22321\end{aligned}$$

   (b) What is the expected number of boxes that contain more than 3 books?
   [*Hint*: Write the number of boxes that contain more than 3 books as a sum of simpler random variables.]

   Answer:
   ≈ 2.0089

   Solution: Let Y denote the number of boxes that contain more than 3 books. Then, we can write Y as

   $$Y = Y_1 + Y_2 + \cdots + Y_9 \ ,$$

   where Y_i (for $i = 1, 2, \dots, 9$) is a Bernoulli random variable indicating whether box $\#i$ contains more than 3 book, that is,

   $$Y_i = \begin{cases} 1 & \text{if box } \#i \text{ contains more than 3 books,} \\ 0 & \text{otherwise.} \end{cases}$$

   By the linearity of expectation,

   $$\mathbb{E}[Y] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2] + \cdots + \mathbb{E}[Y_9]$$

   By symmetry, $Y_1, Y_2, \dots, Y_9$ all have the same expected values, so we can focus on Y_1. Using the result of the previous part, we have

   $$\mathbb{E}[Y_1] = \mathbb{P}(Y_1 = 1) = \mathbb{P}(\text{box \#1 has more than 3 books}) \approx 0.22321 \ .$$

   Therefore,

   $$\mathbb{E}[Y] \approx \underbrace{0.22321 + 0.22321 + \cdots + 0.22321}_{\text{9 times}} = 9 \times 0.22321 \approx 2.0089$$

3. (0 points) On the scale of 0–5, what is your estimate of your grade on this quiz?