American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 2 (due: Wednesday, February 15)

Suggested (ungraded) warm-up exercises:

- Problems 3.2, 3.5 and 3.8 of Tijms's book.

Problem 1 (Inclusion-exclusion for probabilities)**.** Consider a discrete probability model with a (countable) sample space Ω and a probability measure $\mathbb{P}$, so that for every event $E \subseteq \Omega$, we have $\mathbb{P}(E) = \sum_{\omega\in\Omega} \mathbb{P}(\omega)$.

Use the idea of Problem 2 from Assignment 1 to give a proof of the probabilistic inclusion-exclusion principle for three sets: For every three events $A, B, C \subseteq \Omega$,

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\ .$$

Does your argument fail if A, B, C are not finite?

Problem 2 (Birthday problem)**.** Parts (b) and (c) of Problem 3.12 of Tijms's book. Verify that the approximate formula in part (c) gives the right answer when $c = 365$.

Problem 3 (A game of dice)**.** For Problem 3.18 of Tijms's book, do the following:

(a) Describe a hypothetical simulation for estimating the expected value.

(b) Give a description of the sample space, and define the net win as a random variable.

Problem 4 (A drunkard with a drift)**.** Problem 3.26 of Tijms's book.

Problem 5 (Optimal strategy)**.** A bag contains a red balls and b blue balls. At each step, you pick a ball from the bag at random without replacement. If the ball is blue, you win \$1; otherwise, you lose all the money you have won so far and the game stops. You can stop the game whenever you wish. What is an optimal stopping strategy in order to maximize your expected gain?

Problem 6 (Passenger with misplaced boarding pass)**.** One hundred passengers are entering a plane with 100 seats one after another and taking their seats. Each passenger has a boarding pass with his/her seat number written on it. The first passenger has misplaced his boarding pass, so not knowing his seat number, he takes one of the seats at random. Each of the following passengers attempts to sit in his/her designated seat, but if the seat is occupied, takes a random, unoccupied seat. What is the probability that the last passenger sits in his/her designated seat? (*Hint:* To understand the problem intuitively, represent each seat by a point numbered 1 to 100. If the i-th passenger ends up sitting in seat number j, draw an arrow from point i to point j.) Once you have figured out the answer, try to justify your answer with an appropriate probability model.