American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 3 (due: Wednesday, March 1)

Suggested (ungraded) warm-up exercises:

- Problems 4.1, 4.13, 5.1, 5.4, 5.5 of Tijms's book.
- Compute the variance and standard deviation of a Poisson random variable with parameter λ.
- Compute the variance and standard deviation of a binomial random variable with parameters n and p.
- Verify that the variance of the standard normal distribution is 1.

Problem 1 (Substitution rule for discrete RVs). Let X be a discrete random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a function, and consider a new random variable $Y := g(X)$. By definition, the expected value of Y is $\mathbb{E}[Y] := \sum_y \mathbb{P}(Y = y)y$, where the sum runs over all the possible values of Y. Let us assume that the sum converges absolutely, so that the expected value exists. As discussed in class, the expected value of Y can alternatively be calculated as $\mathbb{E}[Y] = \sum_x \mathbb{P}(X = x)g(x)$, where the sum runs over all the possible values of X. Either prove the latter formula, or provide a heuristic justification for it via the law of large numbers.
[*Hint* (for the proof): Write $\mathbb{P}(Y = y) = \sum_{x:g(x)=y} \mathbb{P}(X = x)$, and change the order of summations.]
[*Hint* (for heuristics via LLN): Recall the reasoning behind the definition of the expected value via the law of large numbers. Group the terms according to the value of X.]

Problem 2 (Comparing binomial, Poisson, and normal distributions). The website [https://homepage.divms.uiowa.edu/~mbognar/] contains calculators/visualizers for common probability distributions (thanks to Matt Bognar).

(a) Compare the graphs of the probability mass functions of the binomial and the Poisson distributions for the following choices of the parameters:

- Bin$(n = 50, p = 0.005)$ vs. Poisson$(\lambda = 0.25)$
- Bin$(n = 50, p = 0.05)$ vs. Poisson$(\lambda = 2.5)$
- Bin$(n = 50, p = 0.5)$ vs. Poisson$(\lambda = 25)$
- Bin$(n = 500, p = 0.05)$ vs. Poisson$(\lambda = 25)$

In each case, compare the means, the variances, and $\mathbb{P}(X \leq k)$ for a few values of k, when X has either of the two distributions. Report your observations in light of the law of rare events.

(b) Compare the graphs of the probability mass function of the binomial distribution and the probability density function of the normal distribution for the following choices of the parameters:

- Bin$(n = 50, p = 0.05)$ vs. N$(\mu = 2.5, \sigma^2 = 2.375)$
- Bin$(n = 50, p = 0.5)$ vs. N$(\mu = 25, \sigma^2 = 12.5)$
- Bin$(n = 500, p = 0.05)$ vs. N$(\mu = 25, \sigma^2 = 23.75)$

In each case, compare the means, the variances, and $\mathbb{P}(X \leq k)$ for a few values of k, when X has either of the two distributions. Report your observations in light of the central limit theorem.

Problem 3 (Rolling dice I).

(a) Problem 4.2 of Tijms's book.

(b) Problem 4.10 of Tijms's book.

Problem 4 (King's treasure). Problem 4.7 of Tijms's book.

Problem 5 (Another birthday problem). Problem 4.18 of Tijms's book.

Problem 6 (Rolling dice II).

(a) (Problem 5.15 of Tijms's book)
What happens to the value of the probability of getting at least r sixes in one throw of $6r$ dice as $r \to \infty$? Justify your answer. [*Hint*: Use the CLT.]
Compare this with your answer to Problem 3a.

(b) What happens to the value of the probability of getting at least r sixes in one throw of $5r$ dice as $r \to \infty$? Justify your answer. [*Hint*: Use Chebyshev's inequality.]
Compare this with your answer to the previous part.

Problem 7 (Constant covariances). Problem 5.10 of Tijms's book (only the first part).

Problem 8 (Traffic accidents). Problem 5.18 of Tijms's book. [*Hint*: Argue that a Poisson random variable would be a good model for the number of accidents during one year in the area. Use the fact that when λ is large, the Poisson distribution with parameter λ can be approximated with a normal distribution. What are the parameters of such a normal distribution?]