American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 4 (due: Wednesday, March 13)

Suggested (ungraded) warm-up exercises:

- Problems 6.5, 6.6 of Tijms's book

Problem 1 (Bowls and balls). Problem 6.10 of Tijms's book.

Problem 2 (To operate or not to operate). Problem 6.12 of Tijms's book.

Problem 3 (Envelopes with money). Problem 6.16 of Tijms's book.

Problem 4 (Expected value: interpretation). Recall the interpretation of the expected value of a random variable X as the approximate average of the values of X in a large number of repeated experiments. For a discrete random variable X, this naturally gave rise, via the empirical law of large numbers, to the definition $\mathbb{E}[X] \coloneqq \sum_x \mathbb{P}(X = x)x$. Give a similar justification for the definition of the expected value of a continuous random variable X with density $f(x)$ as $\mathbb{E}[X] \coloneqq \displaystyle\int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$.

[*Hint*: Use the empirical law of large numbers and the interpretation of the probability density function as an idealized histogram. How would you estimate the average of a list of numbers if you only had access to their histogram?]

Problem 5 (Rolling dice). You have a die. Would you find it plausible that the die is fair if:

(a) You rolled the die 100 times and the sum of the numbers facing upwards turned out to be 332?

(b) You rolled the die 1000 times and the sum of the numbers facing upwards turned out to be 3327?

(c) You rolled the die 10000 times and the sum of the numbers facing upwards turned out to be 33276?

Problem 6 (Expected value: alternative expression).

(a) Let X be a *non-negative* discrete random variable taking values in $\mathbb{N} \coloneqq \{0, 1, 2, \ldots\}$. Prove that $\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$. [*Hint*: Write $x = \sum_{k=0}^{x-1} 1$ and change the order of summation.]

(b) Let X be a *non-negative* continuous random variable. Prove that $\mathbb{E}[X] = \displaystyle\int_{y=0}^{\infty} \mathbb{P}(X > y)\,\mathrm{d}y$.

[*Hint*: Write $x = \int_{y=0}^{x} 1\,\mathrm{d}y$ and change the order of integration.]

(c) Prove that the latter integral formula for the expectation is valid also when X is a non-negative discrete random variable, irrespective of whether X takes its values in $\mathbb{N}$ or not.

(d) Use the latter integral formula to calculate the expected value of an exponential random variable with rate λ. Calculate the expected value of a geometric random variable with parameter p in two ways using the above two formulas.

Problem 7 (Absolute value of a normal RV; drunkard's walk). Let Z be a standard normal random variable, and $D \coloneqq |Z|$.

(a) Show that D is a continuous random variable and find its probability density function.

(b) Calculate the expected value of D. [*Remark*: You can either use the distribution of D derived in the previous part or the continuous analogue of the substitution rule examined in the previous assignment.]

(c) Consider a drunkard performing a symmetric random walk down a street, moving at each step either one step forward or one step backward, each with a probability of $1/2$. Let D_n denote the distance of the drunkard from its starting point after n steps. Argue, as in Section 5.8.1 of Tijms's book, that $\mathbb{E}[D_n] \approx \sqrt{\frac{2}{\pi} n}$ when n is large.

(optional) (d) The precise statement of the approximate identity in the previous part is $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\,\mathbb{E}[D_n] = \sqrt{\frac{2}{\pi}}$. This precise phrasing makes it easier to spot a gap in the argument. Does the convergence of probabilities given by the central limit theorem imply the convergence of the expected values?

Problem 8 (Normal distribution). In class, we defined a *normal* random variable as a random variable of the form $X = \alpha Z + \beta$, where Z is a standard normal random variable and $\alpha, \beta \in \mathbb{R}$ are constants with $\alpha \neq 0$. An alternative definition is given in Section 5.1.2 of Tijms's book.

(a) Verify that $\mathbb{E}[X] = \beta$ and $\mathbb{V}\mathrm{ar}[X] = \alpha^2$.

(b) Show that X is a continuous random variable and find its probability density function.
[*Hint*: Given an interval $[a, b]$, write $\mathbb{P}(a \leq X \leq b)$ in terms of Z.]

(c) Explain why your answer to the previous part does not depend on the sign of α.

(d) Prove that the two definitions are equivalent. [*Hint*: In order to prove the equivalence of the two definition, you have to prove that if X satisfies one, it also satisfies the other and vice versa.]