

American University of Beirut
MATH/STAT 233: Probability

2022–2023 Spring

Midterm 1 (Tuesday, February 21, 17:30 – Nicely 212)

How it works: You receive the questions in advance. At the beginning of the exam, you declare which of the questions you have managed to solve. For each question, one student (from those who have solved it) will be called to show their solution on the board.

What is permitted and what is not permitted: Before the exam, you can (and are encouraged to) discuss the problems with the other students in class but not with external people. Getting help from internet forums is also not permitted. Feel free to use your class notes, books, computers, or Wikipedia.

Problem 1 (Coin flipping I). We flip a coin n times in a row.

- (a) Assuming $n = 2023$ and the coin is fair, what is the probability that the number of heads is even?
- (b) How does your answer to the previous question change if $n = 2024$?
- (c) Suppose that $n = 2023$ and the coin has parameter $p = 2/3$. Argue that the probability that the number of heads is even will be very very close (though not exactly equal) to the case in which the coin is fair.

Problem 2 (Coin flipping II). We have a coin with parameter $p \in (0, 1)$, and we flip it n times. Let X denote the number of heads. For which value of k is the probability $\mathbb{P}(X = k)$ maximized?

[Hint 1: Compare $\mathbb{P}(X = k)$ and $\mathbb{P}(X = k + 1)$.]

[Hint 2: Depending on p , there may be one or two solutions.]

Problem 3 (Jar with balls I). A jar contains 13 blue balls and 17 red balls. We draw a ball from the jar at random, record its color, and replace it back into the jar along with 2 extra balls of the same color. Then, we draw another ball from the jar at random.

- (a) What is the chance that the second ball is blue?
- (b) Suppose we repeat the first step of the experiment over and over. In other words, we repeat drawing balls from the jar; each time, we record the color of the drawn ball and then put it back into the jar along with 2 extra balls of the same color. What is the chance that the 20th ball we draw is blue?

[Hint: Consider what happens after one step and use induction.]

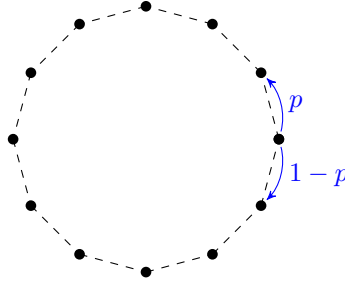
Problem 4 (Jar with balls II). You are offered the following game with a jar that contains 10 blue balls and 2 red balls. You draw a random sample of k balls from the jar without replacement, where the value k can be chosen by you. If all the balls you drew are blue, you win k dollars. Otherwise, you lose 1 dollar.

- (a) What is your expected net gain for any choice of k ?
- (b) What is the optimal choice for k ?

Problem 5 (Oranges in boxes). You distribute 25 oranges over 10 boxes at random.

- (a) Identify the distribution of the number of oranges in box #1.
- (b) What is the probability that box #1 receives more than 3 oranges?
- (c) What is the expected number of boxes that contain more than 3 oranges?

Problem 6 (Particle on a ring). A particle is moving on a ring with n points.



At every step, the particle jumps at random, either one step counter-clockwise or one step clockwise, with probabilities p and $1-p$ respectively ($0 < p < 1$). What is the probability that the particle visits all the n points on the ring before returning to its starting position? [Hint: Compare with the gambler's ruin problem.]

Problem 7 (Misprints in a book). Books from a certain publisher contain on average 0.25 misprints per page.

- (a) Explain why a Poisson random variable would be a reasonable model for the number of misprints in a given page, and identify the parameter of such a Poisson random variable.
- (b) What is the probability that a given page contains at least 5 misprints?
- (c) What is the expected number of misprints in a 400-page book?
- (d) What is the probability that in a 400-page book, there is at least one page with at least 5 misprints?