Assignment 3

Modern Theory of Markov Chains

Due: 04.03.2014

1 (Markov property)**.** Which of the following sequences is a Markov chain? Prove your claims. For each that is a Markov chain, identify the transition probabilities.

a) The sequence $|W_1|, |W_2|, \ldots$, where $W_1, W_2, \ldots$ is a simple random walk on $\mathbb{Z}$.

b) A simple random walk $R_1, R_2, \ldots$ on $\mathbb{Z}^2$ with the additional restriction that the walk cannot pass any position more than once.

c) The sequence $X_1, X_2, \ldots$ where $X_i \triangleq |\{Z_1, Z_2, \ldots, Z_i\}|$ and $Z_1, Z_2, \ldots,$ are independent uniformly distributed random variables in $\{1, 2, \ldots, n\}$.

2 (Existence and uniqueness)**.** This problem outlines simple algebraic proofs of the existence and uniqueness theorems of the finite-state Markov chains. Let K be a finite stochastic matrix.

a) Show that 1 is an eigenvalue of K.

b) Let u be a left eigenvector for K corresponding to eigenvalue 1. Show that $|u|$ is also an eigenvector for K corresponding to eigenvalue 1, where $|u|$ denotes the vector whose elements are the absolute values of the elements of u.
Hint: Show that $|u| \leq |u|\, K$ and consider the inequality obtained by summing over the coordinates.

c) (Existence) Show that K has a stationary distribution.

Next, suppose that K is irreducible.

d) (Positivity) Show that any stationary distribution for K is positive.

e) Let u be a non-zero vector satisfying $uK = u$. Show that the elements of u are either all positive or all negative. Hint: Consider $u + |u|$.

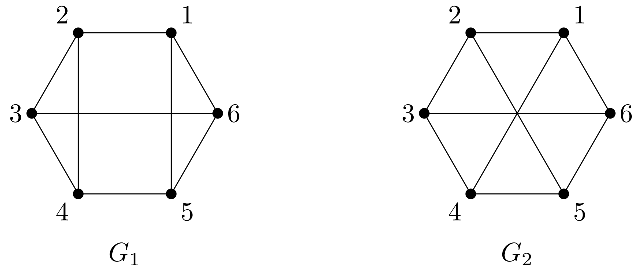
f) (Uniqueness) Show that K has at most one stationary distribution.

(BONUS) **3** (Uniform tightness)**.** As we have seen, a Markov chain with an infinite state space does not need to have a stationary distribution. In the spirit of the topological proof of the existence for finite-state Markov chains, we have the following necessary and sufficient condition for the existence in case of infinite-state chains.

Consider a Markov chain with a possibly infinite countable state space S and transition matrix K. Prove that the Markov chain has a stationary distribution if and only if there is a state $a \in S$ satisfying the following condition: for every $\varepsilon > 0$, there is a finite set $A_\varepsilon \subseteq S$ such that $K^t(a, A_\varepsilon) \geq 1 - \varepsilon$ for all $t \geq 0$.
Remark: When S is infinite, there are several choices of topology on $[0, 1]^S$. In particular, unlike in the case where S was finite, the product topology (i.e., the topology in which

$u_n \to u$ if and only if $u_n(x) \to u(x)$ for all $x \in S$) and the topology of the ℓ^1 norm (i.e., $\|u\| \triangleq \sum_{x\in S} |u(x)|$) are not the same. The product topology is still compact (by Tychonoff's theorem) and metric. With respect to the product topology, the set of probability distributions on S is not closed, but you could verify that the transformation $\pi \mapsto \pi K$ (for a stochastic matrix K) is still continuous when restricted to the set of probability distributions.

4 (Asymptotic behaviour)**.** Consider a simple random walk on each of the following graphs.



a) Using a pseudorandom number generator, make three independent realizations of the walk on G_1 from time 0 to 20 starting from vertex 1, and plot the results as a function of time. Repeat this for the walk on G_2.

b) Now generate 200 independent realizations of the walk on G_1 from time 0 to 50 starting from vertex 1. For each time step $0 \le t \le 50$, calculate the empirical distribution of the position of these 200 walks, and plot this empirical distribution as a function of time. (If you are using Mathematica, you can, for example, use the `ArrayPlot` function.) Repeat this for the walk on G_2. How do you explain the qualitative difference between the results of the experiments on G_1 and G_2?

c) Generate a single realization of the walk on G_1 from time 0 to $N = 700$, starting from vertex 1. For each time $0 \le t \le N$, calculate the density of the time steps $0 \le s \le t$ in which the walk has been in position 1 (or $2, 3, \dots, 6$). Plot this density as a function of time t. Does this density seem to converge to a constant? If necessary, vary the length of the experiment N to see the trend. Repeat the above for the walk on G_2.

d) Pick your favorite function $f : \{1, 2, \dots, 6\} \to \mathbb{R}$. As before, generate a single realization of the walk on either G_1 or G_2 from time 0 to $N = 700$, starting from vertex 1. For each time $0 \le t \le N$, calculate the average value of f over the position of the walk in times 0 up to t. Plot this average value as a function of time t. Does this average value seem to converge to a constant? Vary N to see the trend.

e) Can you explain the observation in part (d) using the observation in part (c)?