Assignment 5

Modern Theory of Markov Chains

Due: 25.03.2014

1 (Aperiodicity)**.** This exercise is meant to review the concept of aperiodicity. Like irreducibility, aperiodicity of states in a Markov chain $(X_t)_t$ depends only on the underlying graph of probable transitions and not on the actual transition probabilities. The *period* of a state a, denoted by $d(a)$, is the greatest common divisor of the lengths of the closed walks starting from a. (If there is no closed walk starting from a, we leave $d(a)$ undefined.) A state is *aperiodic* if it has period 1.

a) Show that every state of a random walk on a connected bipartite graph has period 2.

b) Show that every two states that are strongly connected (i.e., communicate) have the same period.

c) Show that for every aperiodic state a, there is a number $l_a \geq 0$ such that for every $t \geq l_a$, $\mathbf{P}(X_t = a \mid X_0 = a) > 0$. You may use part (d) as a lemma.

d) Let L be a non-empty subset of positive integers that is closed under addition (i.e., $a + b \in L$ whenever $a, b \in L$), and suppose that the greatest common divisor of the elements of L is 1. Prove that $\mathbb{Z}^+ \setminus L$ is finite. (Hint: Pick an arbitrary element $p \in L$, and show that L contains p consecutive numbers $m, m+1, \ldots, m+p-1$.)

e) Show that for any finite-state, irreducible, aperiodic chain with transition matrix K, there is a number l such that $K^l > 0$ (i.e., all the entries of K are strictly positive). Note that if K^l is positive, so is K^t for every $t \geq l$.

2 (Laziness does not distort the equilibrium states)**.** Recall that a necessary condition for the convergence of an irreducible Markov chain to stationary distribution is that the chain is aperiodic. One way to make sure that a Markov chain used in a Monte Carlo chain is aperiodic is to add “laziness” to states: at any step, we flip a (fixed, but possibly biased) coin. If it comes up heads, we make a transition according to the original chain; if it comes up tails, we stay in stay in the current state.

a) Verify that in a lazy version of a Markov chain, every state is aperiodic.

b) Show that a lazy version of a Markov chain has the same stationary distributions as the original one.

3 (A limit theorem for finite groups)**.** Let $(\mathbb{G}, +, 0)$ be a finite commutative group and $p : \mathbb{G} \to [0, 1]$ a probability distribution on $\mathbb{G}$. Let $Z_1, Z_2, \ldots$ be a sequence of independent random variables all with distribution p, and for $n \geq 1$, define $S_n \triangleq Z_1 + Z_2 + \cdots + Z_n$.

a) First, suppose that p is strictly positive. Show that as $n \to \infty$, the distribution of S_n converges to the uniform distribution on $\mathbb{G}$.

b) Next, assume only that $p(0) > 0$. Show that as $n \to \infty$, the distribution of S_n converges to the uniform distribution on a subgroup of $\mathbb{G}$. What additional condition should p satisfy so that the limiting distribution is uniform over $\mathbb{G}$?

(BONUS) c) Can you find a necessary and sufficient condition for the convergence of S_n to the uniform distribution on $\mathbb{G}$?

4 (Bi-infinite chains)**.**

a) Prove that every irreducible and aperiodic bi-infinite Markov chain on a finite state set is in equilibrium.

(BONUS) b) Can you find an example of an irreducible and aperiodic bi-infinite Markov chain with countable state set that is not in equilibrium? (This would be an example of a so-called *symmetry-breaking*: while the transition probabilities are independent of an observer's time reference, the distribution of the chain is not!)