

Assignment 6

Modern Theory of Markov Chains

Due: 1.04.2014

- 1 (Recurrence). Show that every finite-state Markov chain has at least one recurrent state.
- 2 (Drunkard's limit distribution). Consider our old example of a drunkard doing a random walk on a street with n blocks, moving from corner to corner until he reaches his home (corner 0) or the pub (corner n).
 - a) What are the stationary distributions of the drunkard's walk?
 - b) (Exercise 1.17 of the textbook) Generalize this to express the set of stationary distributions of a finite-state Markov chain in terms of the stationary distributions of its irreducible components (= non-transient communicating class).
 - c) Suppose the drunkard starts at corner k . What will be the asymptotic distribution of the drunkard's position? What will be the asymptotic distribution if his starting point is random with distribution μ ?
 - d) Generalize this and describe the asymptotic distribution of a finite-state Markov chain based on the stationary distributions of the irreducible components and the probabilities of hitting each component depending on the initial state. Assume that each irreducible component is aperiodic.
 - e) Let $(X_t)_t$ be a Markov chain with finite state space S and $f : S \rightarrow \mathbb{R}$ an arbitrary function. What can you say about the convergence of the time averages $\frac{f(X_0) + f(X_1) + \dots + f(X_{n-1})}{n}$ as $n \rightarrow \infty$?
- 3 (Random walk recurrence). Prove that a simple random walk on $\mathbb{Z}$ with probability p of going one step to the right and probability $1 - p$ of going one step to the left is recurrent if and only if $p = 1/2$. In case of transience, calculate the probability of escape to infinity. (Hint: consider a drunkard's walk in which the drunkard is drifting towards the pub, and calculate the probability of ending up in the pub if the drunkard starts at position k . Let k be fixed and send the pub to infinity.)
- 4 (Super-harmonic functions). Let $(X_t)_t$ be a Markov chain with state set S and transition matrix K . A function $f : S \rightarrow \mathbb{R}$ is said to be *harmonic* if $Kf = f$ and *super-harmonic* if $Kf \leq f$. (Recall the notation: $(Kf)(x) \triangleq \sum_y K(x, y)f(y) = \mathbf{E}[f(X_{t+1}) \mid X_t = x]$.)
 - a) Verify that if f is super-harmonic, so is Kf , and in particular, $K^t f$ is non-increasing. Conclude that for a non-negative super-harmonic function f , the point-wise limit $f^\infty \triangleq \lim_{t \rightarrow \infty} K^t f$ exists and is harmonic.
 - b) Let $T_a^+ \triangleq \inf\{t > 0 : X_t = a\}$. Show that $f(x) \triangleq \mathbf{P}_x(T_a^+ < \infty)$ is super-harmonic. Furthermore, argue that f is harmonic if and only if a is a recurrent state.

The *martingale convergence theorem* implies that for every non-negative super-harmonic function f , the sequence $f(X_0), f(X_1), \dots$ converges almost surely (i.e., with probability 1). Note that this strengthens the conclusion of part (a), which merely said that the expected value of the elements of this sequence converge.

- c) Prove that an irreducible Markov chain is recurrent if and only if every non-negative super-harmonic function is constant. You may use the above consequence of the martingale convergence theorem.
- d) Use (c) to give an alternate proof that a simple random walk on $\mathbb{Z}$ is recurrent if and only if it is symmetric.
- (BONUS) e) Prove that the simple (symmetric) random walk on a rootless infinite binary tree (i.e., a infinite connected graph with no cycles in which every vertex is connected to three other vertices) is transient.