The University of British Columbia

Final Examination - Dec 15, 2015

Mathematics 300

Closed book examination Time: 2.5 hours

Last Name ____________ **First** __________ **Signature** ____________

Student Number ____________

Special Instructions:

No notes or calculators are allowed. Answer all questions on the sheets provided. Use the backs of the sheets and blank sheets if necessary. Show your final answer clearly by drawing a box around it.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		14
10		6
Total		100

PROBLEM 1. Write the following expressions in the form $a + bi$. Simplify.

a. (5 pts)

$$\frac{1+i}{(1-i)^7}$$

b. (5 pts)

$$[Log(-e^{\pi})]^6$$

PROBLEM 2. Find all z that satisfy the given equation.

a. (5 pts)

$$|e^{\frac{1}{z-1}}| = 1$$

b. (5 pts)

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = i.$$

PROBLEM 3. **a.** (5 pts) Find the radius of convergence of the power series

$$\sum_{k=0}^{\infty}(-1)^k\frac{e^{i\pi k}}{(2-3i)^k}(z-i)^k.$$

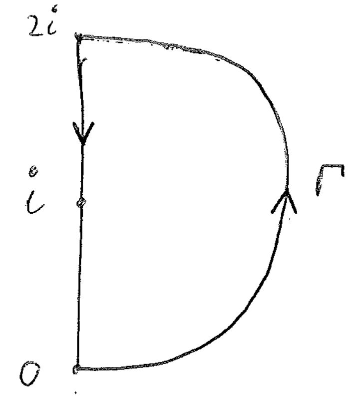
b. (5 pts) Find the annulus in which the following Laurent series converges.

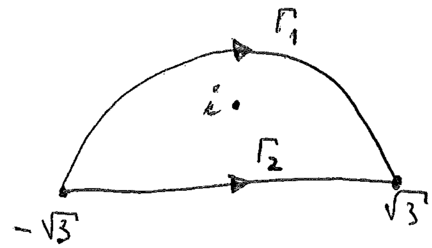
$$\sum_{k=0}^{\infty}\frac{z^k}{(2i)^k}+\sum_{k=1}^{\infty}\frac{1}{(3+i)^k z^k}.$$

PROBLEM 4. (10 pts) Let $Re(z)$ be the real part of z, $Re(x+iy)=x$. Find

$$\int_\Gamma Re(z)dz,$$

where Γ is the curve sketched below.





PROBLEM 5. Let

$$f(z) = \frac{1}{z^2+1},$$

and let Γ_1 and Γ_2 be the two contours shown.

a. (5 pts) Explain the relationship between $\int_{\Gamma_1} f(z)dz$ and $\int_{\Gamma_2} f(z)dz$.

b. (5 pts) Find the integral $\int_{\Gamma_1} f(z)dz$.
(Hint: it may be simpler to find $\int_{\Gamma_2} f(z)dz$ and use part a.)

PROBLEM 6. (10 pts) Classify the singularities of

$$\cos\frac{1}{z+1}+\frac{\sin z - z}{z^3+z^2}.$$

For each singularity you have to either write down a Laurent series expansion or explain why the Laurent series has a certain form.

PROBLEM 7. Let

$$f(z) = \frac{3z-1}{(z-2)(z+3)}.$$

a. (4 pts) Find the partial fraction decomposition of $f(z)$.

b. (6 pts) Find a Laurent series of $f(z)$ that converges for $|z| > 3$. Give the first 4 nonzero terms in the series.

PROBLEM 8. Consider

$$f(z) = \frac{\sin z}{1+z^2}.$$

a. (5 pts) Find the fifth derivative $f^{(5)}(0)$. (Hint: use Taylor series.)

b. (5 pts) Find the residue

$$Res(\frac{1}{f(z)}, \pi).$$

PROBLEM 9. Find the following integrals. In both cases the contour of integration is a circle of radius 2 around 0 traversed counterclockwise once.

a. (7 pts)

$$\int_{|z|=2} \frac{e^z - 1}{z^3 + z^2} dz.$$

b. (7 pts)

$$\int_{|z|=2} \frac{1}{z \sin^2 z} dz.$$

PROBLEM 10. **a.** (3 pts) Let $f(z)$ have only isolated singularities. Assume that $f(z)$ has a Laurent series expansion

$$f(z) = \ldots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots.$$

that converges in the annulus $3 < |z| < 5$. Explain why

$$a_{-1} = \sum Res(f, z_k),$$

where the sum runs over all singular points z_k that lie in the disk $|z| \leq 3$.

b. (3 pts) Let $f(z)$ be an analytic function and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ the Laplacian. Then

$$\nabla^2\left(|f(z)|^2\right) = C|f'(z)|^2$$

for some constant C. Find this constant C.

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