MATH 300 (Complex Variables)

Solutions to Practice Exam

(Final exam of Fall 2015)

Problem 1.

a. We have

$$\frac{1+\mathrm{i}}{(1-\mathrm{i})^7} = \frac{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{(\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}})^7} = \frac{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{(\sqrt{2})^7\mathrm{e}^{-\mathrm{i}\frac{7\pi}{4}}} = \frac{\mathrm{e}^{\mathrm{i}\frac{8\pi}{4}}}{(\sqrt{2})^6} = \boxed{\frac{1}{8}} = \frac{1}{8} + 0\mathrm{i}\ .$$

b. We have $\mathrm{Log}(-\mathrm{e}^{\pi}) = \mathrm{Log}(\mathrm{e}^{\pi}\mathrm{e}^{\mathrm{i}\pi}) = \pi + \mathrm{i}\pi = \pi(1+\mathrm{i})$, therefore

$$[\mathrm{Log}(-\mathrm{e}^{\pi})]^6 = \pi^6(1+\mathrm{i})^6 = \pi^6\big(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big)^6 = \pi^6 2^3 \mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} = \boxed{-8\pi^6\mathrm{i}} = 0 + (-8\pi^6)\mathrm{i}\ .$$

Problem 2.

a. That $\left|\mathrm{e}^{\frac{1}{z-1}}\right| = 1$ means that $\mathrm{e}^{\frac{1}{z-1}}$ is on the unit circle. This is the case if and only if $1/(z-1)$ is purely imaginary. Note that $1/(z-1)$ is purely imaginary if and only if $z-1$ is purely imaginary, which in turn is the case if and only if the real part of z is 1. Therefore, z satisfies the identity in question if and only if its real part is 1.

b. We have

$$\begin{aligned}
\frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}} = \mathrm{i} \qquad & \text{if and only if} \qquad \mathrm{e}^z - \mathrm{e}^{-z} = \mathrm{i}\mathrm{e}^z + \mathrm{i}\mathrm{e}^{-z} \\
& \text{if and only if} \qquad \mathrm{e}^z(1-\mathrm{i}) = \mathrm{e}^{-z}(1+\mathrm{i}) \\
& \text{if and only if} \qquad \mathrm{e}^{2z} = \frac{1+\mathrm{i}}{1-\mathrm{i}} = \mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} \\
& \text{if and only if} \qquad 2z = \frac{\pi}{2}\mathrm{i} + 2k\pi\mathrm{i} \quad \text{for some } k \in \mathbb{Z} \\
& \text{if and only if} \qquad \boxed{z = \frac{\pi}{4}\mathrm{i} + k\pi\mathrm{i} \quad \text{for some } k \in \mathbb{Z}}\ .
\end{aligned}$$

Problem 3.

a. Observe that this is in fact a geometric series, because

$$\sum_{k=0}^{\infty}(-1)^k\frac{\mathrm{e}^{\mathrm{i}\pi k}}{(2-3\mathrm{i})^k}(z-\mathrm{i})^k = \sum_{k=0}^{\infty}\left[\frac{-\mathrm{e}^{\mathrm{i}\pi}}{2-3\mathrm{i}}(z-\mathrm{i})\right]^k\ .$$

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Therefore, the series converges if and only if its ratio parameter has absolute value less than 1. The absolute value of the ratio parameter is

$$\left|\frac{-e^{i\pi}}{2-3i}(z-i)\right| = \left|\frac{z-i}{2-3i}\right| = \frac{1}{\sqrt{13}}\,|z-i|\ ,$$

which is less than 1 if and only if $|z-i| < \sqrt{13}$. Therefore, the radius of the convergence of the series is $\boxed{\sqrt{13}}$. (Alternatively, you could apply the ratio test.)

b. In order for the series to converge, both its tails (the one with positive powers of z and the one with negative powers of z) must converge.

The first series is geometric with parameter $z/(2i)$. Therefore, it converges if and only if $|z/(2i)| < 1$, that is, if and only if $|z| < 2$.

The second series is also geometric, this time with parameter $1/\big[(3+i)z\big]$. Hence, it converges precisely when $\big|1/\big[(3+i)z\big]\big| < 1$, that is, when $|z| > 1/\,|3+i| = 1/\sqrt{10}$.

The combined series converges if and only if both these conditions are satisfied. Therefore, the annulus of convergence is $\boxed{1/\sqrt{10} < |z| < 2}$.

Problem 4. Let Γ_1 denote the semi-circle segment of the contour and Γ_2 denote the vertical segment of the contour. We have

$$\int_\Gamma \operatorname{Re}(z)\,dz = \int_{\Gamma_1} \operatorname{Re}(z)\,dz + \int_{\Gamma_2} \operatorname{Re}(z)\,dz\ .$$

Observe that $\operatorname{Re}(z) = 0$ on Γ_2, hence the second integral is zero. To compute the first integral, we parametrize it as $z(t) = i + e^{it}$ with $t \in [-\pi/2, \pi/2]$. Then, $\operatorname{Re} z(t)$ and $dz = ie^{it}\,dt$. Therefore,

$$\begin{aligned}
\int_\Gamma \operatorname{Re}(z)\,dz = \int_{\Gamma_1} \operatorname{Re}(z)\,dz &= \int_{-\pi/2}^{\pi/2} \cos(t) i e^{it}\,dt \\
&= \frac{i}{2}\int_{-\pi/2}^{\pi/2} (e^{it} + e^{-it})e^{it}\,dt \\
&= \frac{i}{2}\int_{-\pi/2}^{\pi/2} \big[e^{2it} + 1\big]\,dt \\
&= \frac{i}{2}\left[\frac{1}{2i}e^{2it} + t\right]_{t=-\pi/2}^{\pi/2} \\
&= \frac{i}{2}\,[0+\pi] = \boxed{\frac{\pi}{2}i}\ .
\end{aligned}$$

Interesting fact: In general, if γ is a simple, closed, positively-oriented contour, then

$$\int_\gamma \operatorname{Re}(z)\,dz = i \times \langle \text{area enclosed by } \gamma \rangle\ .$$

Can you show this?

Problem 5.

a. Observe that $f(z)$ is everywhere analytic except at two singularities $\pm i$ (both simple poles). If Γ denotes the contour obtained by first going through Γ_2 and then through Γ_1 in the opposite direction, then Γ is a simple, closed, positively-oriented contour surrounding the pole i but not $-\mathrm{i}$. Therefore,

$$\int_{\Gamma_2} f(z)\,\mathrm{d}z - \int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \times \operatorname{Res}(f;\mathrm{i})$$

Since i is a simple pole, we have

$$\operatorname{Res}(f;\mathrm{i}) = \lim_{z\to\mathrm{i}} (z-\mathrm{i}) f(z) = \lim_{z\to\mathrm{i}} \frac{z-\mathrm{i}}{z^2+1} = \lim_{z\to\mathrm{i}} \frac{1}{z+\mathrm{i}} = \frac{1}{2\mathrm{i}} = -\frac{\mathrm{i}}{2}\ .$$

Therefore,

$$\int_{\Gamma_2} f(z)\,\mathrm{d}z - \int_{\Gamma_1} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \times \frac{-\mathrm{i}}{2} = \pi\ ,$$

or equivalently,

$$\boxed{\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z - \pi}\ .$$

b. Following the hint, we compute the integral over Γ_2. Since Γ_2 is simply a segment of the real axis, the integral over Γ_2 is actually a real integral:

$$\int_{\Gamma_2} f(z)\,\mathrm{d}z = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{x^2+1}\,\mathrm{d}x = \Big[\tan^{-1} x\Big]_{x=-\sqrt{3}}^{\sqrt{3}} = \frac{\pi}{3} - \frac{-\pi}{3} = \frac{2\pi}{3}\ .$$

Therefore, using the first part,

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \frac{2\pi}{3} - \pi = \boxed{-\frac{\pi}{3}}\ .$$

Problem 6. The function

$$f(z) := \cos\frac{1}{z+1} + \frac{\sin z - z}{z^2(z+1)}$$

has two singularities: one at $z=-1$ and one at $z=0$.

First, let us argue that the <u>singularity at $z=0$ is removable</u>. To see this, note that $\sin z - z$ has (everywhere convergent) Taylor expansion

$$\begin{aligned}\sin z - z &= -\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ &= z^3\left(-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right)\end{aligned}$$

which means $\sin z - z$ has a zero of order 3 at $z=0$. Therefore,

$$\begin{aligned}\lim_{z\to0} f(z) &= \cos 1 + \lim_{z\to0} \frac{z^3\left(-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right)}{z^2(z+1)} \\ &= \cos 1 + \lim_{z\to0} \frac{z\left(-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right)}{z+1} = \cos 1\ ,\end{aligned}$$

which implies that the singularity is removable. (Alternatively, you could use l'Hôpital, or any other method you like, to show that the limit exists.)

Next, we argue that the singularity at $z = -1$ is essential. Observe that the second term has a simple pole at $z = -1$, because

$$\frac{\sin z - z}{z^2(z+1)} = \frac{(\sin z - z)/z^2}{z+1}$$

and $(\sin z - z)/z^2$ is analytic and non-zero at $z = -1$. In particular, the Laurent expansion of this term about $z = -1$ (in the annulus $0 < |z+1| < 1$) has the form

$$\frac{\sin z - z}{z^2(z+1)} = a_{-1}(z+1)^{-1} + a_0 + a_1(z+1) + a_2(z+1)^2 + \cdots$$

where $a_{-1} \neq 0$. For the first, term we start with the (everywhere convergent) Taylor expansion

$$\cos w = 1 - \frac{1}{2!}w^2 + \frac{1}{4!}w^4 - \frac{1}{6!}w^6 + \cdots$$

and substitute w with $(z+1)^{-1}$ to obtain the Laurent expansion

$$\cos\frac{1}{z+1} = 1 - \frac{1}{2!}(z+1)^{-2} + \frac{1}{4!}(z+1)^{-4} - \frac{1}{6!}(z+1)^{-6} + \cdots ,$$

which is convergent as long as $z \neq -1$. Combining the two Laurent expansions, we see that in the Laurent expansion of $f(z)$ about $z = -1$ (in the annulus $0 < |z+1| < 1$) there are infinitely many negative values k (namely all even $k < 0$) for which the coefficient of $(z+1)^k$ is non-zero. Hence the singularity is essential.

Problem 7.

a. The partial fraction decomposition of $f(z)$ has the form

$$\frac{3z-1}{(z-2)(z+3)} = \frac{A}{z-2} + \frac{B}{z+3} .$$

To find A, we multiply both sides by $z-2$, cancel the common terms, and set $z = 2$:

$$\frac{3z-1}{z+3} = A + B\cdot\frac{z-2}{z+3} \qquad \Longrightarrow \qquad \boxed{A = 1} .$$

Similarly, to find B, we multiply both sides by $z+3$, cancel the common terms, and set $z = -3$:

$$\frac{3z-1}{z-2} = A\cdot\frac{z+3}{z-2} + B \qquad \Longrightarrow \qquad \boxed{B = 2} .$$

Therefore,

$$\frac{3z-1}{(z-2)(z+3)} = \boxed{\frac{1}{z-2} + \frac{2}{z+3}} .$$

b. In order to find the Laurent expansion centered at $z = 0$ for $|z| > 3$, we try to extract a convergent geometric series from each term of the partial fraction decomposition. The first term can be expanded as

$$\frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-2z^{-1}} = z^{-1}\sum_{k=0}^{\infty}(2z^{-1})^k = \sum_{\ell=1}^{\infty}2^{\ell-1}z^{-\ell} ,$$

which is valid, because $\left|2z^{-1}\right| = 2/|z| < 1$ when $|z| > 3$. Similarly, the second term can be expanded as

$$\frac{2}{z+3} = \frac{2}{z} \cdot \frac{1}{1+3z^{-1}} = 2z^{-1} \sum_{k=0}^{\infty} (-3z^{-1})^k = \sum_{\ell=1}^{\infty} 2(-3)^{\ell-1} z^{-\ell} ,$$

which is valid, because $\left|3z^{-1}\right| = 3/|z| < 1$ when $|z| > 3$. Combining the two expansions, we get

$$\begin{aligned} f(z) &= \boxed{\sum_{\ell=1}^{\infty} \left[2^{\ell-1} + 2(-3)^{\ell-1}\right] z^{-\ell}} \\ &= 3z^{-1} - 4z^{-2} + 22z^{-3} - 46z^{-4} + \cdots \end{aligned}$$

when $|z| > 3$.

Problem 8.

a. Following the hint, we start by finding the Taylor expansions of the two factors of $f(z)$ about $z = 0$. We have

$$\begin{aligned} \sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots && \text{(valid for all } z \in \mathbb{C}) \\ \frac{1}{1+z^2} &= 1 - z^2 + z^4 - z^6 + z^8 - \cdots && \text{(valid when } \left|-z^2\right| < 1) \end{aligned}$$

The Taylor expansion of $f(z)$ about $z = 0$ is the Cauchy product of the above expansion. In particular, the coefficient of z^5 in the product is

$$1 + \frac{1}{3!} + \frac{1}{5!}$$

which must coincide with $f^{(5)}(0)/5!$. Therefore,

$$\begin{aligned} f^{(5)}(0) &= \boxed{5!\left(1 + \frac{1}{3!} + \frac{1}{5!}\right)} \\ &= 5! + \frac{5!}{3!} + 1 = 5 \times 4 \times 3 \times 2 \times 1 + 5 \times 4 + 1 = \boxed{141} . \end{aligned}$$

b. In order to find the residue of

$$\frac{1}{f(z)} = \frac{1+z^2}{\sin z}$$

at $z = \pi$, we first note that $1/f(z)$ has a *simple* pole at $z = \pi$. To see this, observe that $1 + z^2$ does not have a zero/pole at $z = \pi$, while $\sin z$ has a zero of order 1 at $z = \pi$ (because $\sin \pi = 0$ but $(\sin z)'|_{z=\pi} = \cos \pi \neq 0$). Therefore, $\sin z = (z - \pi)g(z)$ for some function $g(z)$ that is analytic and non-zero at $z = \pi$. Hence,

$$\frac{1}{f(z)} = \frac{1+z^2}{\sin z} = \frac{(1+z^2)/g(z)}{z-\pi} ,$$

where $(1 + z^2)/g(z)$ is analytic and non-zero at $z = \pi$.

Since $1/f(z)$ has a simple pole at $z = \pi$, we have

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{f(z)};\pi\right) &= \lim_{z\to\pi}(z-\pi)\frac{1+z^2}{\sin z} \\
&= (1+\pi^2)\lim_{z\to\pi}\frac{z-\pi}{\sin z} \\
&= (1+\pi^2)\lim_{\zeta\to 0}\frac{\zeta}{\sin(\zeta+\pi)} \\
&= -(1+\pi^2)\lim_{\zeta\to 0}\frac{\zeta}{\sin\zeta} \\
&= \boxed{-(1+\pi^2)}\,.
\end{aligned}$$

(Alternatively, we could use l'Hôpital to find the above limit.)

Problem 9.

a. *Approach 1 (via residue theorem).* The integrand (which we will call $f(z)$) has two singularities at $z = 0$ and $z = -1$, both of which are in the interior of the the contour.

The singularity at $z = -1$ is a simple pole, because

$$\frac{\mathrm{e}^z - 1}{z^2(z+1)} = \frac{(\mathrm{e}^z-1)/z^2}{z+1}$$

where $(\mathrm{e}^z - 1)/z^2$ is analytic and non-zero at $z = -1$. The residue at $z = -1$ is

$$\operatorname{Res}(f;-1) = \lim_{z\to-1}(z+1)f(z) = \lim_{z\to-1}\frac{\mathrm{e}^z-1}{z^2} = \boxed{\mathrm{e}^{-1}-1}\,.$$

The singularity at $z = 0$ is also a simple pole, despite the z^2 in the denominator. Namely, $\mathrm{e}^z - 1 = z + z^2/2! + \cdots$ has a zero of order 1 at $z = 0$, hence $\mathrm{e}^z - 1 = zg(z)$ for some function $g(z)$ that is analytic and non-zero at $z = 0$, hence

$$\frac{\mathrm{e}^z-1}{z^2(z+1)} = \frac{zg(z)}{z^2(z+1)} = \frac{g(z)/(z+1)}{z}$$

where $g(z)/(z+1)$ is analytic and non-zero at $z = 0$. Therefore,

$$\begin{aligned}
\operatorname{Res}(f;0) = \lim_{z\to-1} zf(z) &= \lim_{z\to-1}\frac{\mathrm{e}^z-1}{z(z+1)} \\
&= \left.\frac{\mathrm{e}^z}{2z+1}\right|_{z=0} && \text{(by l'Hôpital)} \\
&= \boxed{1}\,.
\end{aligned}$$

Combining the two residues, we get

$$\oint_{|z|=2}\frac{\mathrm{e}^z-1}{z^2(z+1)}\,\mathrm{d}z = 2\pi\mathrm{i}\left[\operatorname{Res}(f;-1)+\operatorname{Res}(f;0)\right] = 2\pi\mathrm{i}\left[\mathrm{e}^{-1}-1+1\right] = \boxed{\frac{2\pi}{\mathrm{e}}\mathrm{i}}\,.$$

Approach 2 (breaking the fraction and using Cauchy's formula). Using partial fraction decomposition, we find

$$\frac{1}{z^2(z+1)} = \frac{-1}{z} + \frac{1}{z^2} + \frac{1}{z+1}\,.$$

Therefore, if we set $h(z) := \mathrm{e}^z - 1$, then

$$\frac{\mathrm{e}^z - 1}{z^2(z+1)} = \frac{-h(z)}{z} + \frac{h(z)}{z^2} + \frac{h(z)}{z+1}\,.$$

Since $h(z)$ is analytic everywhere (in particular, on and inside the contour) and both $z = 0$ and $z = -1$ are in the interior of the contour, we can use Cauchy's formula to get

$$\begin{aligned}\oint_{|z|=2} \frac{\mathrm{e}^z - 1}{z^2(z+1)}\,\mathrm{d}z &= -\oint_{|z|=2} \frac{h(z)}{z}\,\mathrm{d}z + \oint_{|z|=2} \frac{h(z)}{z^2}\,\mathrm{d}z + \oint_{|z|=2} \frac{h(z)}{z+1}\,\mathrm{d}z \\ &= -2\pi\mathrm{i} \times h(0) + 2\pi\mathrm{i} \times h'(0) + 2\pi\mathrm{i} \times h(-1) \\ &= 2\pi\mathrm{i}\big[-(1-1) + 1 + (\mathrm{e}^{-1} - 1)\big] \\ &= \boxed{2\pi\mathrm{e}^{-1}\mathrm{i}}\,.\end{aligned}$$

b. *Approach 1 (standard).* The integrand has infinitely many singularities (at $z = k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$) but only $z = 0$ is in the interior of the contour. We first identify the type of this singularity. Since $\sin z$ has a simple zero at $z = 0$, $(\sin z)^2$ has a zero of order 2 and $z(\sin z)^2$ has a zero of order 3 at $z = 0$. Therefore, $f(z) := 1/\big[z(\sin z)^2\big]$ has a pole of order 3 at $z = 0$. Hence, the Laurent expansion of $f(z)$ about $z = 0$ (valid for $0 < |z| < \pi$) has the form

$$f(z) = a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z^1 + \cdots\,.$$

In order to find a_{-1} (the residue at $z = 0$), we multiply by z^3 to get

$$z^3 f(z) = \underbrace{a_{-3} + a_{-2}z + a_{-1}z^2 + a_0 z^3 + a_1 z^4 + \cdots}_{g(z)}$$

and note that a_{-1} is the coefficient of z^2 in the Taylor expansion of $g(z)$. Therefore,

$$\operatorname{Res}(f;0) = a_{-1} = \frac{g^{(2)}(0)}{2!} = \frac{1}{2!}\lim_{z\to 0}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{z^2}{(\sin z)^2}\right]_{z=0}\,.$$

Finding the residue in this fashion requires lengthy (but doable) computation.

Approach 2 (formal manipulations). An alternative way to find the residue at $z = 0$ is to try to find (the first few terms of) the Laurent expansion about $z = 0$ via formal manipulations. We start with

$$\begin{aligned}\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ (\sin z)^2 &= (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots) \\ &= z^2 + \frac{-1}{3}z^4 + \frac{2}{45}z^6 + \cdots \\ &= z^2(1 + \underbrace{\frac{-1}{3}z^2 + \frac{2}{45}z^4 + \cdots}_{h(z)})\end{aligned}$$

which are convergent for every value of z, because $\sin z$ and $(\sin z)^2$ are entire. Note that $h(z) \to 0$ as $z \to 0$. In particular, $|h(z)| < 1$ in a sufficiently small neighbourhood of the origin. Therefore, we have the expansion

$$\begin{aligned}\frac{1}{z(\sin z)^2} &= z^{-3}\frac{1}{1+h(z)}\\ &= z^{-3}\big(1-h(z)+h(z)^2-h(z)^3+\cdots\big)\\ &= z^{-3}\left(1-\left[\frac{-1}{3}z^2+\frac{2}{45}z^4+\cdots\right]+\left[\frac{1}{9}z^4+\cdots\right]+\left[\frac{-1}{27}z^6+\cdots\right]+\cdots\right)\\ &= z^{-3}\left(1+\frac{1}{3}z^2+\frac{1}{15}z^4+\cdots\right)\\ &= z^{-3}+\frac{1}{3}z^{-1}+\frac{1}{15}z+\cdots\end{aligned}$$

for z in a small neighbourhood of $z=0$. Observe that although the expansion involves many Cauchy products, we only need to do a small number of calculations in order to find the coefficient of z^{-1}, which is $1/3$. We conclude that $\operatorname{Res}\big(1/[z(\sin z)^2]\big)=1/3$. Therefore,

$$\oint_{|z|=2}\frac{1}{z(\sin z)^2}\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname{Res}\left(\frac{1}{z(\sin z)^2}\right)=\boxed{\frac{2\pi}{3}\mathrm{i}}\,.$$

Problem 10.

a. Let γ be the circle $|z|=4$ oriented counter-clockwise. Then, the coefficient a_{-1} of the Laurent expansion of $f(z)$ in $3<|z|<5$ (centered at $z=0$) can be written as

$$a_{-1}=\frac{1}{2\pi\mathrm{i}}\int_\gamma f(\zeta)\,\mathrm{d}\zeta\,.$$

(In case you do not remember this, this follows from direct term-by-term integration of the series.) On the other hand, the residue theorem says that

$$\int_\gamma f(\zeta)\,\mathrm{d}\zeta = 2\pi\mathrm{i}\sum\operatorname{Res}(f;z_k)$$

where the sum runs over the singular points z_k that lie in the interior of γ. Since the function is analytic in $3<|z|<5$, the singularities in the interior of γ are all within the closed disk $|z|\le 3$. It follows that

$$a_{-1}=\frac{1}{2\pi\mathrm{i}}\int_\gamma f(\zeta)\,\mathrm{d}\zeta=\sum\operatorname{Res}(f;z_k)$$

where the sum runs over the singularities of $f(z)$ that lie in the closed disk $|z|\le 3$.

b. Let u and v denote the real and imaginary parts of f, so that $f=u+\mathrm{i}v$. Then, $|f(z)|^2=u^2+v^2$. We have

$$\begin{aligned}\frac{\partial}{\partial x}(u^2+v^2) &= 2uu_x+2vv_x\\ \frac{\partial^2}{\partial x^2}(u^2+v^2) &= 2(u_x)^2+2uu_{xx}+2(v_x)^2+2vv_{xx}\end{aligned}$$

and similarly,

$$\frac{\partial^2}{\partial y^2}(u^2+v^2)=2(u_y)^2+2uu_{yy}+2(v_y)^2+2vv_{yy}\,.$$

Therefore,

$$\begin{aligned}
\nabla^2 |f(z)|^2 &= \nabla^2(u^2+v^2) \\
&= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2) \\
&= 2(u_x)^2 + 2uu_{xx} + 2(v_x)^2 + 2vv_{xx} + 2(u_y)^2 + 2uu_{yy} + 2(v_y)^2 + 2vv_{yy} \\
&= 2\left[(u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2 + u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy})\right] \; .
\end{aligned}$$

Recall that the real and imaginary parts of an analytic function are harmonic, that is, $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$. Hence,

$$\nabla^2 |f(z)|^2 = 2\left[(u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2\right] \; . \qquad (\spadesuit)$$

On the other hand, the derivative of an analytic function can be written in either of the following two ways

$$f'(z) = u_x + \mathrm{i}v_x = v_y - \mathrm{i}u_y \; .$$

Therefore,

$$\left|f'(z)\right|^2 = (u_x)^2 + (v_x)^2 = (v_y)^2 + (u_y)^2 \; . \qquad (\clubsuit)$$

Combining ($\spadesuit$) and ($\clubsuit$), we get

$$\nabla^2 |f(z)|^2 = 4\left|f'(z)\right| \; .$$

Therefore, $\boxed{C = 4}$.