Full Name: ____________________

Student No: ____________________

Grade:

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1. (2 points)

(a) Find the 4th roots of $-8+8\sqrt{3}\,\mathrm{i}$ and write them in the form $a+b\,\mathrm{i}$.

Solution: Note that $-8+8\sqrt{3} = 16\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. Therefore,

$$\begin{aligned}(-8+8\sqrt{3})^{\frac{1}{4}} &= \{\sqrt[4]{16}\,\mathrm{e}^{\mathrm{i}\frac{1}{4}(\frac{2\pi}{3}+2k\pi)} : k\in\mathbb{Z}\}\\ &= \{2\mathrm{e}^{\mathrm{i}(\frac{\pi}{6}+\frac{k\pi}{2})} : k\in\mathbb{Z}\}\\ &= \{2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}},\quad 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}},\quad 2\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}},\quad 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}\}\\ &= \{\boxed{\sqrt{3}+\mathrm{i}},\quad \boxed{-1+\mathrm{i}\sqrt{3}},\quad \boxed{-\sqrt{3}-\mathrm{i}},\quad \boxed{1-\mathrm{i}\sqrt{3}}\}\end{aligned}$$

(b) Compute the integral $\displaystyle\int_0^\pi (\sin\theta)^2\cos 2\theta\,\mathrm{d}\theta$.
(HINT: Write sin and cos in terms of the exponential function and simplify.)

Solution: We have

$$\begin{aligned}(\sin\theta)^2\cos 2\theta &= \left(\frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}\right)^2\left(\frac{\mathrm{e}^{\mathrm{i}2\theta}+\mathrm{e}^{-\mathrm{i}2\theta}}{2}\right)\\ &= -\frac{1}{8}(\mathrm{e}^{\mathrm{i}2\theta}+\mathrm{e}^{-\mathrm{i}2\theta}-2)(\mathrm{e}^{\mathrm{i}2\theta}+\mathrm{e}^{-\mathrm{i}2\theta})\\ &= -\frac{1}{8}(\mathrm{e}^{\mathrm{i}4\theta}+1+1+\mathrm{e}^{-\mathrm{i}4\theta}-2\mathrm{e}^{\mathrm{i}2\theta}-2\mathrm{e}^{-\mathrm{i}2\theta})\\ &= -\frac{1}{4}\cos 4\theta+\frac{1}{2}\cos 2\theta-\frac{1}{4}\,.\end{aligned}$$

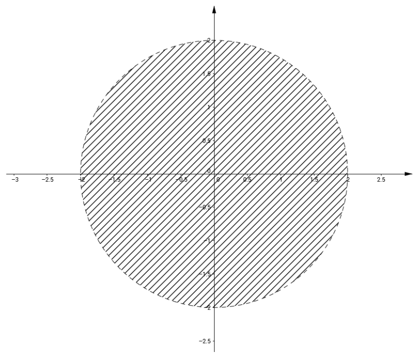
Therefore,

$$\int_0^\pi (\sin\theta)^2\cos 2\theta\,\mathrm{d}\theta = \int_0^\pi \left[\frac{-1}{4}\cos 4\theta+\frac{1}{2}\cos 2\theta-\frac{1}{4}\right]\mathrm{d}\theta = \boxed{-\frac{\pi}{4}}$$

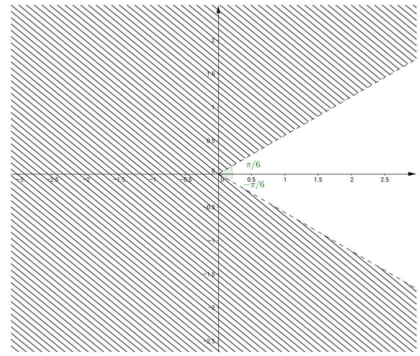
2. (5 points) Let A be the open disk of radius 2 centered at the origin, B the set of points $z \in \mathbb{C}$ such that $|\operatorname{Arg} z| > \pi/6$, and $C := \{z \in \mathbb{C} : \operatorname{Re} z \geq 1\}$. Let D be the intersection of A and B, and let E be the intersection of C and D.

   (a) Sketch the three sets A, B, and C on the complex plane and clarify the boundary points.

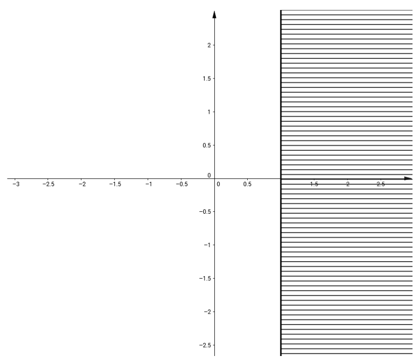
Solution: Solid lines are included, dashed lines are excluded.



A



B



C

   (b) Is D a domain?

   (A *domain* is a set that is both open and connected.)

Answer: Yes

   (c) Is E a region?

   (A *region* is a domain together with some, none or all of its boundary points.)

Answer: No (not connected)

3. (3 points) Show (by means of a geometric or algebraic argument) that if u and v are two distinct points on the unit circle, then the real part of $\dfrac{u+v}{u-v}$ must be zero.

Solution:
Algebraic argument:
Multiplying the numerator and the denominator by the conjugate of the denominator, we have

$$\begin{aligned}\frac{u+v}{u-v} &= \frac{u+v}{u-v}\cdot\frac{\overline{u}-\overline{v}}{\overline{u}-\overline{v}}\\ &= \frac{u\overline{u}-u\overline{v}+v\overline{u}-v\overline{v}}{|u-v|^2}\\ &= \frac{\cancel{|u|^2}-\cancel{|v|^2}+v\overline{u}-\overline{v\overline{u}}}{|u-v|^2}\\ &= \frac{2\mathrm{i}\,\mathrm{Im}(v\overline{u})}{|u-v|^2}\,,\end{aligned}$$

which is purely imaginary.

Geometric argument:
Call A, B, C the points represented by u, $u+v$ and $u-v$ respectively, and denote the origin by O. The two triangles $\triangle OAB$ and $\triangle OAC$ are isosceles, because $|OA| = |AB| = |AC| = 1$. Therefore, $\angle OBA = \angle AOB$ and $\angle COA = \angle ACO$. Since the sum of the latter four angles is π (or $-\pi$, depending on the orientation), we find that $\angle COB = \angle COA + \angle AOB = \pi/2$ (or $-\pi/2$, depending on the orientation). This means that $u+v$ is perpendicular to $u-v$. It follows that the principal argument of $\dfrac{u+v}{u-v}$ is either $\pi/2$ of $-\pi/2$, which means its real part must be zero.