Full Name: ______________________

Student No: ______________________

Grade:

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1. (4 points) Determine whether each of the following statements is TRUE or FALSE. [No explanation is needed.]

   (a) The complex exponential function maps vertical lines to rays starting at (but not containing) the origin, and it maps the horizontal lines to circles centered at the origin.

   Answer: FALSE

   Solution: It is the other way around. A vertical line $x = x_0$ is mapped to a circle of radius e^{x_0} centered at the origin, whereas a horizontal line $y = y_0$ is mapped to the ray starting at (but not containing) the origin with angle y_0.

   (b) The function $f(x + \mathrm{i}y) := y + \mathrm{i}x$ is nowhere differentiable.

   Answer: TRUE

   Solution: If f is differentiable at a point $z = x + \mathrm{i}y$, so is $g(z) := -\mathrm{i}f(z) = x - \mathrm{i}y = \overline{z}$. But we know that $g(z) = \overline{z}$ is nowhere differentiable.

   Alternatively, $\partial u/\partial y = 1$ and $\partial v/\partial x = 1$, hence the Cauchy–Riemann relation $\partial u/\partial y = -\partial v/\partial x$ is nowhere satisfied.

   (c) If $f(z)$ is analytic with $f'(z) = 0$ in an open set A (not necessarily connected), then f is necessarily constant in A.

   Answer: FALSE

   Solution: The statement would be true if A is a *connected* open set (i.e., a domain) but not if it is not connected. For instance,

$$f(z) := \begin{cases} 1 & \text{if } \operatorname{Re} z > 1, \\ 0 & \text{if } \operatorname{Re} z < -1, \end{cases}$$

   is analytic with $f'(z) = 0$ over the open set $\{z : |\operatorname{Re} z| > 1\}$.

   (d) The function $u(x, y) := \mathrm{e}^{-y} \sin x$ is harmonic in the entire plane.

   Answer: TRUE

   Solution: You can verify that u satisfies the Laplace equation. Alternatively, you may observe that u is the imaginary part of the entire function $f(z) := \mathrm{e}^{\mathrm{i}z}$.

2. (4 points)

   (a) Find all the points $z = x + \mathrm{i}y$ at which the function

$$f(z) := x^2y - y + x + \mathrm{i}(xy^2 + y + 1)$$

   is differentiable.

   (b) Identify where f is analytic.

Solution:

(a) The partial derivatives of $u(x,y) := x^2y - y + x$ and $v(x,y) := xy^2 + y + 1$ are

$$\frac{\partial u}{\partial x} = 2xy + 1\ , \qquad \frac{\partial u}{\partial y} = x^2 - 1\ ,$$
$$\frac{\partial v}{\partial x} = y^2\ , \qquad \frac{\partial v}{\partial y} = 2xy + 1\ .$$

Since the partial derivatives exist and are *continuous* everywhere, the function f is differentiable precisely where the Cauchy–Riemann relations are satisfied, that is, where

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\ .$$

The first relation is always satisfied. The second is satisfied if $x^2 - 1 = -y^2$, that is, if $x^2 + y^2 = 1$. Therefore, f is differentiable on the unit circle centered at the origin and nowhere else.

(b) The circle $x^2 + y^2 = 1$ contains no open disk (its interior is empty), so f is nowhere analytic.

3. (2 points) Show that if $f(z)$ is analytic in the entire plane and its image is contained in the hyperbola $\{(x, y) : xy = 1\}$, then f has to be constant.

Solution: If $f = u+iv$ is analytic in the entire plane, it satisfies the Cauchy–Riemann relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ .$$

everywhere. If f maps the entire plane into the hyperbola $\{(x, y) : xy = 1\}$, then $uv = 1$ everywhere. Taking partial derivative with respect to x and y gives

$$\begin{cases} \dfrac{\partial u}{\partial x} v + u \dfrac{\partial v}{\partial x} = 0 \ , \\ \dfrac{\partial u}{\partial y} v + u \dfrac{\partial v}{\partial y} = 0 \ , \end{cases} \tag{$\diamondsuit$}$$

Combined with the Cauchy–Riemann equations (writing everything in terms of the partial derivatives of u), we have

$$\begin{cases} \dfrac{\partial u}{\partial x} v - u \dfrac{\partial u}{\partial y} = 0 \ , \\ \dfrac{\partial u}{\partial y} v + u \dfrac{\partial u}{\partial x} = 0 \ , \end{cases} \tag{$\clubsuit$}$$

Since $uv = 1$, neither u nor v is ever 0. It follows (e.g., by solving ($\clubsuit$) for $\partial u/\partial x$ and $\partial u/\partial y$) that $\partial u/\partial x = \partial u/\partial y = 0$ in the entire plane. Using the Cauchy–Riemann relations, we also get $\partial v/\partial x = \partial v/\partial y = 0$ in the entire plane. Therefore, u and v are both constant (note: the plane is open and connected). We conclude that $f = u + iv$ is also constant.