Full Name: ____________________

Student No: ____________________

Grade:

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1. (4 points) Determine whether each of the following statements is TRUE or FALSE. [No explanation is needed.]

   (a) $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ for every $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

   Answer: FALSE

   Solution: The imaginary part of $\operatorname{Log} z$ (i.e., $\operatorname{Arg} z$) always lies in the interval $(-\pi, \pi]$, but the sum of two numbers in $(-\pi, \pi]$ may fall off the interval. For example, if $z_1 = z_2 = -1$, then $\operatorname{Log} z_1 = \operatorname{Log} z_2 = \pi\mathrm{i}$, but $\operatorname{Log} z_1 z_2 = \operatorname{Log} 1 = 0$.

   (b) $\mathrm{e}^{\operatorname{Log} z} = z$ for every $z \in \mathbb{C} \setminus \{0\}$, and $\operatorname{Log} \mathrm{e}^z = z$ for every $z \in \mathbb{C}$.

   Answer: FALSE

   Solution: The first part is true by the definition of the logarithm, but the identity $\operatorname{Log} \mathrm{e}^z = z$ holds only for $z \in \{x + \mathrm{i}y : y \in (-\pi, \pi]\}$. Note that the exponential function is not one-to-one on the entire plane, and the imaginary part of Log is (by definition) always in $(-\pi, \pi]$. However, for every $z \in \mathbb{C}$, we can find $k \in \mathbb{Z}$ such that $\operatorname{Log} \mathrm{e}^z = z + \mathrm{i}2k\pi$.

   (c) The coefficient of $1/z$ in the partial fraction decomposition of $R(z) := \dfrac{z-1}{z^2(z+1)}$ is 2.

   Answer: TRUE

   Solution: The partial fraction decomposition of $R(z)$ has the form

   $$\frac{z-1}{z^2(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1}$$

   In order to find A, we first multiply both sides by z^2 (and cancel), differentiate both sides with respect to z and set $z = 0$. This gives $A = 2$.

   (d) If $z \neq 0$ and $\alpha \in \mathbb{R}$, then all the values of z^α are on a circle centered at the origin, and all the values of $z^{\mathrm{i}\alpha}$ are on a ray emitting from the origin.

   Answer: TRUE

   Solution: By definition,

   $$z^\alpha = \mathrm{e}^{\alpha \log z} = \mathrm{e}^{\alpha(\ln|z| + \mathrm{i} \arg z)} = \left[\mathrm{e}^{\alpha \ln|z|}\right] \mathrm{e}^{\mathrm{i}(\alpha \arg z)} .$$

   Since $\alpha \ln|z|$ and all the values of $\alpha \arg z$ are real, it follows that all the values of z^α are on the circle with radius $\mathrm{e}^{\alpha \ln|z|}$ centered at the origin.

Similarly,

$$z^{\mathrm{i}\alpha} = \mathrm{e}^{\mathrm{i}\alpha \log z} = \mathrm{e}^{\mathrm{i}\alpha(\ln|z| + \mathrm{i}\arg z)} = \left[\mathrm{e}^{-\alpha \arg z}\right] \mathrm{e}^{\mathrm{i}(\alpha \ln|z|)} .$$

from which it can be seen that all the values of $z^{\mathrm{i}\alpha}$ are on the ray with angle $\alpha \ln|z|$ emitting from the origin.

2. (2 points) Argue that $u(x,y) \coloneqq (\ln|z|)^2 - (\operatorname{Arg} z)^2$ (where $z = x + \mathrm{i}y$) is harmonic in the domain $D^* \coloneqq \mathbb{C} \setminus \{x + \mathrm{i}y : x \le 0 \text{ and } y = 0\}$.
[HINT: Recall that the real and imaginary parts of an analytic function are harmonic.]

Solution: Observe that u is the real part of

$$(\operatorname{Log} z)^2 = (\ln|z| + \mathrm{i}\operatorname{Arg} z)^2 = \big((\ln|z|)^2 - (\operatorname{Arg} z)^2\big) + \mathrm{i}\,(2(\ln|z|)(\operatorname{Arg} z)) \ .$$

Since $f(z) \coloneqq \operatorname{Log} z$ is analytic in D^* and $g(w) \coloneqq w^2$ is analytic everywhere, $(\operatorname{Log} z)^2 = g(f(z))$ is analytic in D (by the chain rule). Therefore, its real part (which is u) is harmonic in D^*.

3. (4 points) Determine the domain of analyticity for $f(z) := \mathrm{Log}(\mathrm{e}^{\mathrm{i}z} + 1)$. Sketch the set of points at which f is not analytic and compute $f'(0)$.

Solution: The function $g(z) := \mathrm{e}^{\mathrm{i}z} + 1$ is analytic everywhere and the function $h(w) := \mathrm{Log}\, w$ is analytic everywhere except on the non-positive real axis

$$N^* := \{a + \mathrm{i}b : a \le 0 \text{ and } b = 0\} .$$

By the chain rule, $f(z) = h(g(z))$ is analytic everywhere except where $g(z) \in N^*$. Hence,

$$\begin{aligned} f(z) \text{ analytic} \quad &\iff \quad \mathrm{e}^{\mathrm{i}z} + 1 \notin N^* \\ &\iff \quad \mathrm{e}^{\mathrm{i}z} \notin -1 + N^* \\ &\iff \quad \mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} \notin -1 + N^* \\ &\iff \quad \mathrm{e}^{-y+\mathrm{i}x} \notin -1 + N^* \end{aligned}$$

with $z = x + \mathrm{i}y$. Note that $-1 + N^*$ is the set of real numbers ≤ -1. Therefore,

$$\begin{aligned} f(z) \text{ not analytic} \quad &\iff \quad \mathrm{e}^{-y+\mathrm{i}x} \text{ is a real} \le -1 \\ &\iff \quad \boxed{y \le 0 \text{ and } x = (2k+1)\pi \text{ for some } k \in \mathbb{Z}} \, . \end{aligned}$$

In words, the domain of analyticity of f is the entire plane excluding the vertical rays emitting downwards from points $(2k + 1)\pi$ (for each $k \in \mathbb{Z}$) on the real axis.

In particular, f is analytic at $z = 0$. By the chain rule,

$$f'(z) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^{\mathrm{i}z} + 1} \, .$$

Therefore, $\boxed{f'(0) = \mathrm{i}/2}$.