Full Name: ______________________

Student No: ______________________

Grade:

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1. (4 points) Determine whether each of the following statements is TRUE or FALSE. [No explanation is needed.]

   (a) $\displaystyle\oint_{|z|=1} \overline{z}\,\mathrm{d}z = \oint_{|z|=1} \frac{1}{z}\,\mathrm{d}z.$

   Answer: TRUE

   Solution: Note that for z on the unit circle, $\overline{z} = \dfrac{1}{z}$.

   (b) If $f(z)$ is analytic at every point in the closed disk $\overline{D} := \{z : |z| \leq 1\}$ and $|f(z)|$ takes a maximum value somewhere in $\overline{D}$, then f is necessarily constant in $\overline{D}$.

   Answer: FALSE

   Solution: For example, the function $f(z) := z$ is analytic at every point in $\overline{D}$ and the maximum of $|f(z)|$ over $\overline{D}$ is achieved at every point on the boundary (the maximum is 1!) but $f(z)$ is not constant.

   (c) If $f(z)$ has an anti-derivative in a domain D, then $f(z)$ is analytic in D.

   Answer: TRUE

   Solution: The anti-derivative of f is analytic in D, and the derivative of an analytic function is also analytic.

   (d) $\displaystyle\int_\gamma \frac{z^{16} - 2016}{z^{2017}}\,\mathrm{d}z = 0$ for every closed contour γ that does not pass through the origin.

   Answer: TRUE

   Solution: The function $f(z) := \dfrac{z^{16} - 2016}{z^{2017}} = z^{-2001} - 2016z^{-2017}$ has an anti-derivative $F(z) := \frac{-1}{2000}z^{-2000} + z^{-2016}$ in the domain $\mathbb{C} \setminus \{0\}$. Therefore, its integrals over closed contours within this domain vanish.

2. (4 points) Find the value of the following integrals.

(a) $\displaystyle\int \frac{\mathrm{Re}(z)}{z}\,\mathrm{d}z$ from $-3\mathrm{i}$ to $3\mathrm{i}$ along the circle $|z| = 3$ traversed counter-clockwise.

Solution: We can use the parametrization $z(t) := 3\mathrm{e}^{\mathrm{i}t}$ where $t \in [-\pi/2, \pi/2]$. Then, $\mathrm{d}z = 3\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t$ and $\dfrac{\mathrm{Re}(z(t))}{z(t)} = \dfrac{\cos t}{\mathrm{e}^{\mathrm{i}t}}$. Therefore,

$$\int \frac{\mathrm{Re}(z)}{z}\,\mathrm{d}z = \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\cancel{\mathrm{e}^{\mathrm{i}t}}}\, 3\,\mathrm{i}\,\cancel{\mathrm{e}^{\mathrm{i}t}}\,\mathrm{d}t = 3\mathrm{i}\int_{-\pi/2}^{\pi/2} \cos t\,\mathrm{d}t = 3\mathrm{i}\,\sin t\Big|_{t=-\pi/2}^{\pi/2} = \boxed{6\mathrm{i}}\,.$$

(b) $\displaystyle\oint_{|z|=5} \frac{2\cos z}{z^2(2z-\pi)}\,\mathrm{d}z$

Solution: Let γ denote the contour $|z| = 5$ oriented counter-clockwise. Note that γ has both poles $z = 0$ and $z = \pi/2$ in its interior. We can either *split the contour* or *split the function*. Let us do the latter. Using partial fraction decomposition, we have

$$\frac{1}{z^2(z-\pi/2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-\pi/2}\,,$$

which gives

$$\int_\gamma \frac{2\cos z}{z^2(2z-\pi)}\,\mathrm{d}z = A\int_\gamma \frac{\cos z}{z}\,\mathrm{d}z + B\int_\gamma \frac{\cos z}{z^2}\,\mathrm{d}z + C\int_\gamma \frac{\cos z}{z-\pi/2}\,\mathrm{d}z\,.$$

Since $\cos z$ is analytic everywhere, in particular on and inside γ, we can apply Cauchy's formula to find each integral, namely

$$\int_\gamma \frac{\cos z}{z}\,\mathrm{d}z = 2\pi\mathrm{i}\cos 0 = 2\pi\mathrm{i}\,,$$
$$\int_\gamma \frac{\cos z}{z^2}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{1!}(-\sin 0) = 0\,,$$
$$\int_\gamma \frac{\cos z}{z-\pi/2}\,\mathrm{d}z = 2\pi\mathrm{i}\cos(\pi/2) = 0\,.$$

Put together, we get

$$\int_\gamma \frac{2\cos z}{z^2(2z-\pi)}\,\mathrm{d}z = A \times 2\pi\mathrm{i}\,.$$

We now calculate A and find $A = -4/\pi^2$. Therefore,

$$\int_\gamma \frac{2\cos z}{z^2(2z-\pi)}\,\mathrm{d}z = \boxed{-\frac{8}{\pi}\mathrm{i}}\,.$$

3. (2 points) Let $f(z)$ be an entire function and suppose that $|f(z)| \leq 1000\,|\mathrm{e}^z|$ whenever $|z| \geq 10$. Show that $f(z) = c\,\mathrm{e}^z$ for some constant $c \in \mathbb{C}$. [HINT: consider $f(z)/\mathrm{e}^z$.]

Solution: Let $g(z) \coloneqq f(z)/\mathrm{e}^z$ and note that $g(z)$ is entire, because both $f(z)$ and e^z are entire and e^z is everywhere non-zero. By the assumption, $|g(z)| \leq 1000$ whenever $|z| \geq 10$. On the other hand, $|g(z)|$ is continuous and hence has a maximum value M on the closed disk $|z| \leq 10$. We do not know the value M, but whatever it is, we have $|g(z)| \leq \max\{M, 1000\}$ for every $z \in \mathbb{C}$. That is, $g(z)$ is bounded. According to Liouville's theorem, since g is a bounded entire function, it has to be constant, that is, $g(z) = c$ for some constant $c \in \mathbb{C}$. Recalling the definition of g, we get $f(z)/\mathrm{e}^z = c$, hence $f(z) = c\,\mathrm{e}^z$ for every z.