Full Name: ______________________

Student No: ______________________

Grade:

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1. (4 points) Determine whether each of the following statements is TRUE or FALSE. [No explanation is needed.]

   (a) If the series $\sum_{k=0}^{\infty} a_k z^k$ converges at $z = 2\mathrm{i}$, then its radius of convergence is at least 2.

   Answer: TRUE

   Solution: Recall that if a series $\sum_{k=0}^{\infty} a_k z^k$ converges at $z = z_1$, it also converges at any point z with $|z| < |z_1|$.

   (b) The Taylor series of e^z about $z = 0$ converges uniformly over the unit disk.

   Answer: TRUE

   Solution: Recall that a power series $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly over a disk of radius R' centered at the origin as long as R' is smaller that its radius of convergence. Since e^z is entire, its Taylor series is convergent in the entire plane, hence the radius of convergence of its Taylor series $\sum_{k=0}^{\infty} \frac{1}{k!} z^k$ is ∞.

   (c) The function $\operatorname{Log} z$ has a Laurent expansion about $z = 0$ that is convergent in the punctured unit disk $0 < |z| < 1$.

   Answer: FALSE

   Solution: If a Laurent series converges in the punctured disk $0 < |z| < 1$, its sum is analytic in the punctured disk. However, the principal logarithm $\operatorname{Log} z$ is not analytic on its branch cut $\{x + \mathrm{i}y : x \leq 0, y = 0\}$, hence it is not analytic everywhere in the punctured disk $0 < |z| < 1$.

   (d) The function $\dfrac{1 - \cos z}{z^3}$ has a pole of order 3 at $z = 0$.

   Answer: FALSE

   Solution: Note that $1 - \cos z$ also has a zero of order 2 at $z = 0$. In particular, the function has Laurent expansion

   $$\frac{1 - \cos z}{z^3} = \frac{1}{z^3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots\right) = \frac{1}{2!}z^{-1} - \frac{1}{4!}z + \frac{1}{6!}z^3 - \cdots$$

   about $z = 0$, which means $z = 0$ is a pole of order 1.

2. (3 points) Find a Laurent expansion of $g(z) := \dfrac{2(z+2)}{z^2-1}$ about $z=-2$ that is convergent at $z=0$.

Solution: The factor $(z+2)$ is already in the right form. For the rest, we start with partial fraction decomposition

$$\frac{2}{z^2-1} = \frac{1}{z-1} + \frac{-1}{z+1} \; .$$

For the first fraction, $z=0$ is within the disk $|z+2|<3$ in which the fraction is analytic, hence we expect a Taylor series. Trying to extract a geometric series, we write

$$\begin{aligned} \frac{1}{z-1} &= \frac{1}{z+2-3} = \frac{-1}{3} \cdot \frac{1}{1-\frac{z+2}{3}} \\ &= \frac{-1}{3} \sum_{k=0}^{\infty} \left(\frac{z+2}{3}\right)^k = -\sum_{k=0}^{\infty} \frac{1}{3^{k+1}} (z+2)^k \; . \end{aligned}$$

which is valid as long as $|z+2|<3$, in particular at $z=0$. For the second term, again we try to extract a geometric series:

$$\begin{aligned} \frac{-1}{z+1} &= \frac{-1}{z+2-1} = \frac{-1}{z+2} \cdot \frac{1}{1-\frac{1}{z+2}} \\ &= \frac{-1}{z+2} \sum_{k=0}^{\infty} \left(\frac{1}{z+2}\right)^k = \frac{-1}{z+2} \sum_{k=0}^{\infty} (z+2)^{-k} = -\sum_{\ell=1}^{\infty} (z+2)^{-\ell} \; , \end{aligned}$$

which is valid as long as $|z+2|>1$, in particular at $z=1$.

Combining these two expansions with the factor $(z+2)$, we get

$$\begin{aligned} \frac{2(z+2)}{z^2-1} &= (z+2)\left[\frac{1}{z-1} + \frac{-1}{z+1}\right] \\ &= (z+2)\left[-\sum_{k=0}^{\infty} \frac{1}{3^{k+1}} (z+2)^k - \sum_{\ell=1}^{\infty} (z+2)^{-\ell}\right] \\ &= \boxed{-\sum_{m=1}^{\infty} \frac{1}{3^m} (z+2)^m - \sum_{n=0}^{\infty} (z+2)^{-n}} \end{aligned}$$

which is valid whenever $1<|z+2|<3$, in particular at $z=0$.

To be sure, we can also verify that at $z=0$, the value of the series agrees with the value of the function (both are 4).

3. (3 points) Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{1+z^{2k}}$ converges if and only if z is *not* on the unit circle.

Solution: Observe that for $|z| < 1$, the denominator of the k-th term is very close to zero for large values of k, hence the series is very close to a convergent geometric series. On the other hand, for $|z| > 1$ and k large, the k-th term is very close to z^{-k}, hence the series is again very close to a convergent geometric series. To make this precise, we examine three cases:

I. Suppose that $|z| < 1$. Then, $\left|1 + z^{2k}\right| \geq 1 - |z|^{2k}$ by the triangular inequality. Note that in this case $|z|^{2k} \to 0$ as $k \to \infty$. In particular, if k is large enough, then $|z|^{2k} < 1/2$. (The choice $\varepsilon = 1/2$ is arbitrary as long as it is larger than 0 and smaller than 1.) It follows that, for k large enough,

$$\left|\frac{z^k}{1+z^{2k}}\right| \leq \frac{|z|^k}{1-|z|^{2k}} < \frac{|z|^k}{1/2} = 2\,|z|^k \ ,$$

that is, the series is dominated by the convergent geometric series $2\sum_{k=0}^{\infty} |z|^k$. Therefore, according to the comparison test, the series must converge.

II. Suppose that $|z| > 1$. Setting $\zeta := 1/z$, we have

$$\sum_{k=0}^{\infty} \frac{z^k}{1+z^{2k}} = \sum_{k=0}^{\infty} \frac{(1/\zeta)^k}{1+(1/\zeta)^{2k}} = \sum_{k=0}^{\infty} \frac{\zeta^k}{\zeta^{2k}+1}$$

which is essentially the same series. In particular, since $|z| > 1$, we have $|\zeta| < 1$, and the result of Case I shows that the series converges.

III. Suppose that $|z| = 1$. Then, $\left|1 + z^{2k}\right| \leq 1 + |z|^{2k} = 2$ by the triangular inequality. Hence,

$$\left|\frac{z^k}{1+z^{2k}}\right| \geq \frac{|z|^k}{2} \geq \frac{1}{2}$$

which means the k-th term of the series does not approach 0 as $k \to \infty$. Therefore, the series does not converge.

In summary, the series diverges when z is on the unit circle and converges otherwise.